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PROCEDURES FOR THE DESIGN OF THERMAL
PROTECTION SYSTEMS FOR MANEUVERABLE
RE-ENTRY VEHICLES

Donald Turrentine

TECHNICAL DOCUMENTARY REPORT NO. ASD-TDR-62-625

September 1962

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Booster and Power Division
Dyna-Soar Engineering Office
Aeronautical Systems Division
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73p. incl illus, tables, 17 refs.

Unclassified Report
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Analytical procedures are presented for determining
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1. Heat transfer
2. Numerical analysis
3. Aerodynamic heating
4. Aircraft structures

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FOREWORD

This report was prepared by the Environmental Control Branch of the Booster and Power Division, Dyna-Soar Engineering Office, Deputy for Engineering, Aeronautical Systems Division, Wright-Patterson Air Force Base, Ohio, under Weapon System 620A.

The work was performed for the University of Buffalo, Buffalo, New York, in partial fulfillment of the requirement for the degree of Master of Science in the field of Mechanical Engineering. Professor Howard E. Strauss served as graduate advisor. The University of Buffalo has granted permission to the USAF to publish and distribute the thesis as an ASD Technical Documentary Report. Only those changes necessary to make the thesis meet the requirements of an ASD Technical Documentary Report have been made.

ABSTRACT

Atmospheric re-entry of earth-orbital, hypersonic glide vehicles creates thermal problems. The heat affects not only the materials and construction of the airframe but also the crew and various subsystems of the vehicle. Successful solution of these problems depends upon the development of an effective thermal protective concept, which will also give the designer some latitude in his design philosophy. The role of the protective system is to significantly attenuate the influx of heat that is aerodynamically generated within the surrounding boundary layer. Attenuation is accomplished by combining external radiation shielding elements with backup insulation materials and an appropriate cooling system.

Analytical procedures are presented for determining significant system parameters by transforming the differential heat conduction or diffusion equation into an algebraic expression by employing the calculus of finite differences. The adaptation of the resulting equation to digital computer programming is discussed, and numerical results are presented to indicate systems of minimum weight.

PUBLICATION REVIEW

This technical documentary report has been reviewed and is approved.

FOR THE COMMANDER:



WILLIAM E. LAMAR
Chief, Dyna-Soar Engineering Office
Deputy for Engineering

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LIST OF SYMBOLS

A_0, A_1, A_2	coefficients of the polynomial $k(u)$
B_0, B_1	coefficients of the polynomial $C_p(u)$
C	heat transfer coefficient, $\frac{\text{BTU}}{(\text{hr}) (\text{sq. ft.}) (^\circ\text{F})}$
C_a	heat transfer coefficient - air space
C_b	heat transfer coefficient - heat shield
C_f	heat transfer coefficient - insulation package
C_h	heat transfer coefficient - film
C_r	heat transfer coefficient - equivalent radiation
c_p	specific heat, $\frac{\text{BTU}}{(\text{lb}) (^\circ\text{F})}$
D_0, D_1, D_2	coefficients of the polynomial $C_d(u)$
ϵ	surface emissivity
E_0, E_1, E_2	coefficients of the polynomial $C_f(u)$
F_a	configuration factor
F_ϵ	emissivity factor
h	distance increment, inches
h_{fg}	enthalpy, $\frac{\text{BTU}}{\text{lb}}$
i, j, m, n	integers
k	thermal conductivity, $\frac{\text{BTU (in.)}}{(\text{hr}) (\text{sq. ft.}) (^\circ\text{F})}$
L	length, inches
p	time increment, minutes
Q	heat per unit area, $\frac{\text{BTU}}{\text{sq. ft.}}$
q	rate of heat transfer per unit area, $\frac{\text{BTU}}{(\text{hr}) (\text{sq. ft.})}$
S	area, sq. ft.
t	time, minutes
U, u	temperature, degrees F

LIST OF SYMBOLS (CONT'D)

W	total system weight, lbs per sq. ft.
w	weight rate of coolant expended, $\frac{\text{lbs}}{(\text{hr})(\text{sq. ft})}$
x, y, z	spatial variables, inches
α	thermal diffusivity, sq. ft. per hr
ρ	density, lbs per cu. ft.
σ	Stefan-Boltzmann constant, $0.173 \times 10^{-8} \frac{\text{BTU}}{(\text{hr})(\text{sq. ft})(^{\circ}\text{F abs.})^4}$

1.0 INTRODUCTION

1.1 General

Although aeronautical technology offers a number of techniques for the successful recovery of rocket-boosted orbital vehicles, the most attractive philosophy is derived from the principles of lifting body re-entry. Unlike trajectories characteristic of ballistic or semiballistic shapes that are controllable only to the extent of orbit ejection sequencing, a lifting body or glide vehicle can be piloted to a preselected site where a conventional landing is made. Atmospheric re-entry descent, within limited parameters, depends only on the judicious use of the kinetic and potential energy of the vehicle.

The maneuvering capability of this type of vehicle during descent can be defined within velocity-altitude limitations of a flight corridor that is bounded by aerodynamic, aeroelastic, structural, and heating considerations. The upper limit is a result of instability of the vehicle at high angles of attack or a maximum attainable aerodynamic lift, while the lower limit is a function of maximum heating rates, allowable acceleration loads, or permissible dynamic pressure. An example of a glide-vehicle flight corridor is shown in Figure 1.

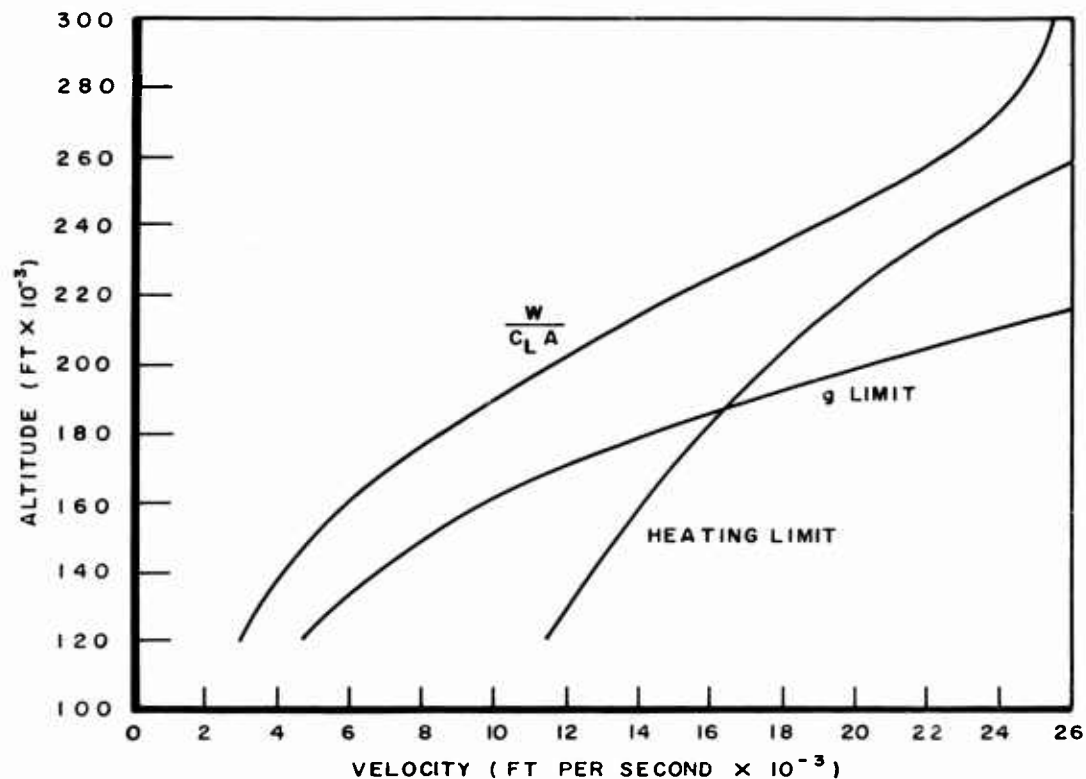


Figure 1. Flight Corridor

Various thermal protection concepts affect both of these limits for the following reasons:

a. Aerodynamic lift is affected by wing loading or by the weight of the thermal protection system.

b. Load carrying capability is a function of structural temperature, which is influenced by the effectiveness of the thermal protection system.

As a result, thermal protection system optimization, that is, adequate protection for minimum weight, contributes to a broader altitude and velocity operational capability.

Glide bodies, unlike ballistic shapes, experience long re-entry times while exposed to relatively low aerodynamic-heating rates. This heat is transferred by boundary-layer convection to the surface of the vehicle where it can be radiated to space, absorbed by the outer skin, and transmitted to internal sections. Considerable advantage might be realized in external dissipation of this convective heat, since the technique does not result in additional vehicle weight, but is a function merely of the emissive power of the outside surface. A knowledge of this factor along with the rate at which heat is aerodynamically generated will enable, by use of the Stefan-Boltzmann relationship, an approximation of the equilibrium temperature of the exterior of the vehicle.

Since the surface equilibrium temperatures during re-entry flight will exceed the internal conditioned compartment temperature, heat will penetrate to areas of compartment location. The modes of transfer, which include solid conduction, radiation, gaseous conduction, and convection, must be minimized to a degree predicated by the relative contribution of each to the total heat influx.

Heat conduction that passes energy directly through the sections of a solid in contact with the hotter surfaces can be lowered by using materials that are classified as poor thermal conductors and by reducing the cross-sectional area normal to the flow path. The latter may be achieved by incorporation of a large number of gaseous spaces, since conduction through a gas is much less than conduction through a solid. Another technique would involve breaking the solid down into a collection of fine particles, which, by forming many points of contact within the material, would result in an increased resistance to the flow of heat.

Heat transfer by radiation from the surface of a hot body occurs through a mechanism similar to electromagnetic wave phenomena and is, in fact, frequency dependent and intimately connected to that of light transmission. Geometric orientation of the surfaces relative to the incident radiation and techniques for reducing surface emissivity both contribute to the attenuation of this process.

The transfer of heat by conduction within a gas is described as diffusion process in which the molecules are in motion between warm and cool areas. In addition, an exchange in kinetic energy occurs as wandering molecules collide with one another. Gases in a free unconfined state exhibit virtually no change in thermal conductivity as a result of pressure changes, because their average path length decreases as the number of molecules increase in proportion to the pressure. On the other hand, confined gases show lower thermal conductivities at reduced pressures because the number of molecules available for transporting heat energy approaches zero. This principle can be applied to reduce the effects of gas conduction by using the ambient pressures existing at altitude during re-entry.

Heat convection takes place as a result of a free-molecular mixing motion initiated by temperature differences within a gas. If the gas is contained within small spaces, the motion of molecules will be confined to an extent negating heat convection.

If during re-entry the temperatures within the pressurized areas of the vehicle are maintained essentially at equilibrium without using compartment conditioning equipment to absorb heat penetrating from the external surface, then this function must be assigned to the thermal protection system. The feasibility of this direction is justified in view of available techniques that are more effective than those that depend on the circulation of a gaseous cooling media within a relatively large volume. A description of the aspects of thermal protection systems is presented in the following section.

1.2 Thermal Protection Concepts

1.2.1 General

Systems protecting the internal structure or compartments from the effects of aerodynamic heating consist of three major integrated elements. First, the exterior surface forming high emissivity heat-shield segments transmits to the surrounding environment a great deal of the heat reaching the vehicle from the boundary layer. The effectiveness of this scheme in minimizing internal heat penetration is shown in Figure 2. The second element, a lightweight thermal insulation, attenuates the remaining heat conducted to areas surrounding the environmentally controlled compartments. At this location, the heat is removed by a cooling system, which serves as the remaining element.

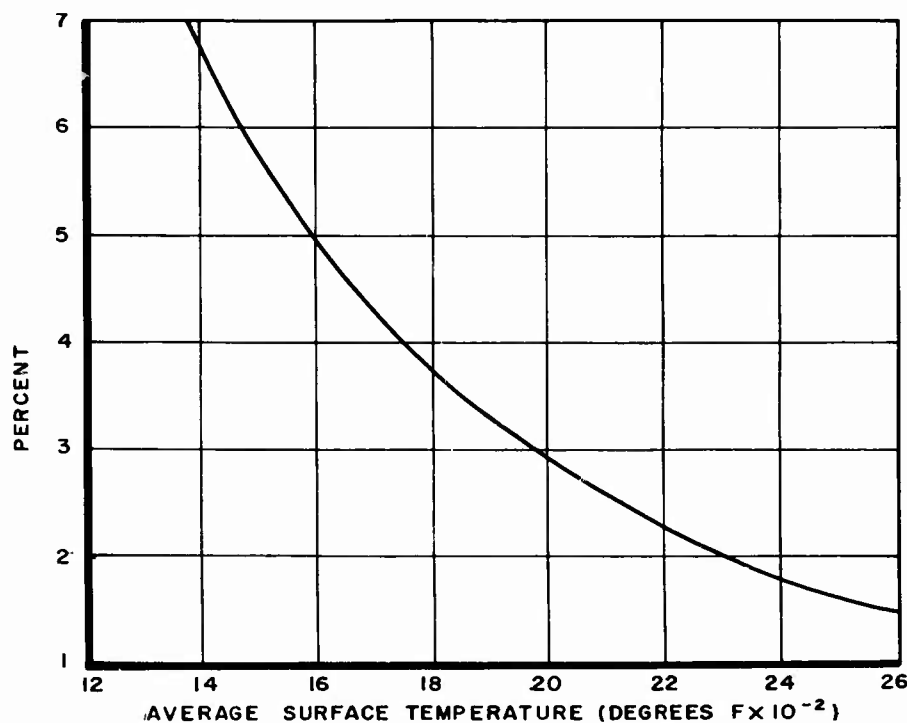


Figure 2. Percent of Total Heat Removed by Cooling System

1.2.2 Radiation and Ablation Cooling

The primary factors associated with the development of radiation heat shields are those governing the selection of materials, emissivity of the external surface, and the evolution of a suitable design configuration. Criteria forming a basis for material selection include oxidation resistance and strength requirements that are compatible with the specified service life and the external environments. As a result of the latter, surface-temperature histories of the vehicle are predictable functions of instantaneous heat generation rates during re-entry and the emissivity of the surface. The relationship between heating rates, emissivity, and heat-shield equilibrium temperature is shown in Figure 3. Details of design configurations are concerned more with the incorporation of stiffness and thermal stress alleviating features into lightweight heat shields to preclude panel buckling.

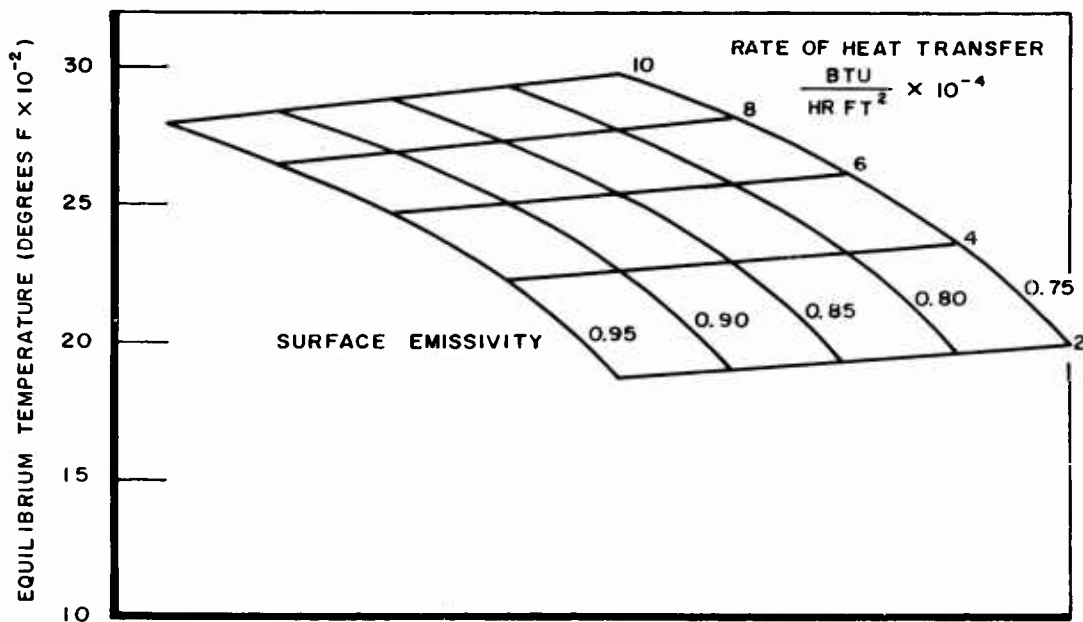


Figure 3. Effect of Surface Emissivity and Heat Transfer Rate on Equilibrium Temperature

Vehicle surfaces that can withstand temperatures up to 2000°F can be constructed with super alloy materials using current manufacturing methods and materials technology. As metallurgical research uncovers new materials, this limit might be extended to 3000°F. Refractory metals, for example, are generally useful to 2700°F if effective oxidation-resistant coatings are available and high surface emissivities can be maintained. Furthermore, refractory metals are not suitable for many structural applications; hence, aerodynamic loads must be transmitted to more ductile members. Other refractory materials such as ceramics and graphites possess excellent elevated temperature characteristics which, when integrated with suitable design concepts, also can be applied to exterior surface panels.

Still another approach is that of using an ablative surface that is combined with radiation shields to give a re-entry vehicle capability for returning from superorbital missions. Under these circumstances, trajectories that result in heating rates that are associated with the ablation process must be considered.

1.2.3 Thermal Insulations and Cooling Systems

1.2.3.1 General

Radiation cooling that provides a high surface emissivity for a low weight, and materials characterized by relatively high heats of ablation, although effective, do not completely solve the problems that are associated with thermal protection for re-entry heating. As a result of extended flight through the earth's atmosphere, temperature gradients are established that cause heat to penetrate the external surface. This heat must be dissipated to maintain environments that are compatible with those required for human comfort and equipment operation.

Several approaches can be considered for removing this heat or confining it to a region between the compartment shell and the outer surface of the vehicle.

The methods that involve either the principles of heat storage or the vaporization of a liquid are as follows:

- a. Thermal insulation systems
- b. Liquid heat sink systems
- c. Systems that combine thermal insulation and liquid heat sinks.

Low thermal diffusivity insulations can delay the influx of aerodynamic heat until completion of the mission. Two obvious physical-property requirements for these materials are a low thermal conductivity and high specific heat at the temperature and pressure levels that result from re-entry trajectories. Insulations available from industry, which can retain their form at elevated temperatures and low pressures do not display values for the properties just mentioned to permit their effective application. Denser forms and thicker sections of the materials must be employed as compensating measures to obtain the desired temperature-time response throughout the system.

Protection systems using the latent heat of vaporization of suitable coolants represent a technological approach for removing boundary-layer heat that penetrates the exterior surface. The total heat removed or the weight of the coolant expanded during a re-entry trajectory is a function of the thermal conductance between the heated and cooled surfaces. A convenient means for reducing the conductance involves the use of insulating materials and form a protection system configuration when combined with a cooling system.

Conformation of the schemes selected for further review and optimization resulted from the use of relationships derived in Reference 4 of the Bibliography. Solutions are presented in Figure 4 as a result of considering step-function temperature inputs that are equivalent to mean surface temperatures associated with re-entry flight. Weights for combined systems were determined from a steady-state heat balance that equated the heat conducted through the insulation to that removed by the coolant. The results verify that the most effective thermal-protection systems for lifting body re-entry vehicles combine insulations with expendable coolants.

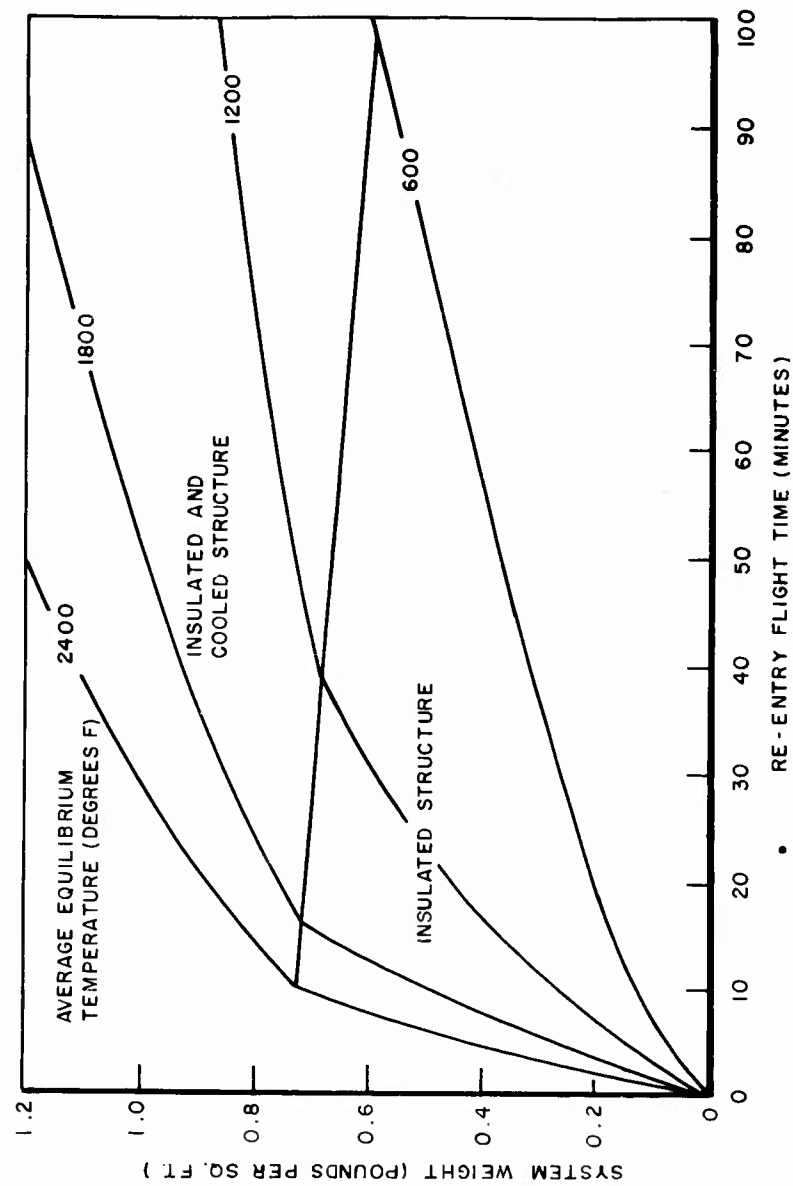


Figure 4. Thermal Protection Configuration for Minimum Weight

1.2.3.2 Insulation Materials

The selection of a thermal insulation involves a comparison of properties that indicates the potential capabilities of the materials. The insulating materials must provide not only a maximum resistance to the flow of heat but they must be light in weight and demonstrate the ability to endure mission environments with no evidence of degradation that is detrimental to service life.

A measure of the relative effectiveness of insulations may be derived by expressing the weight of the material as a function that depends on the physical properties. If effectiveness can be defined as the reciprocal of the weight of insulation, the function will be equal to the inverse of the density-thermal conductivity product. The effects of altitude on this product should be considered when thermal conductivity values are selected.

A comparison of several commercially available, uncontained insulating materials is shown in Figure 5. The possibility exists that material characteristics dictate containment concepts that significantly affect the thermal conductance of the system. For these cases, an apparent thermal conductivity that includes containment effects should be used

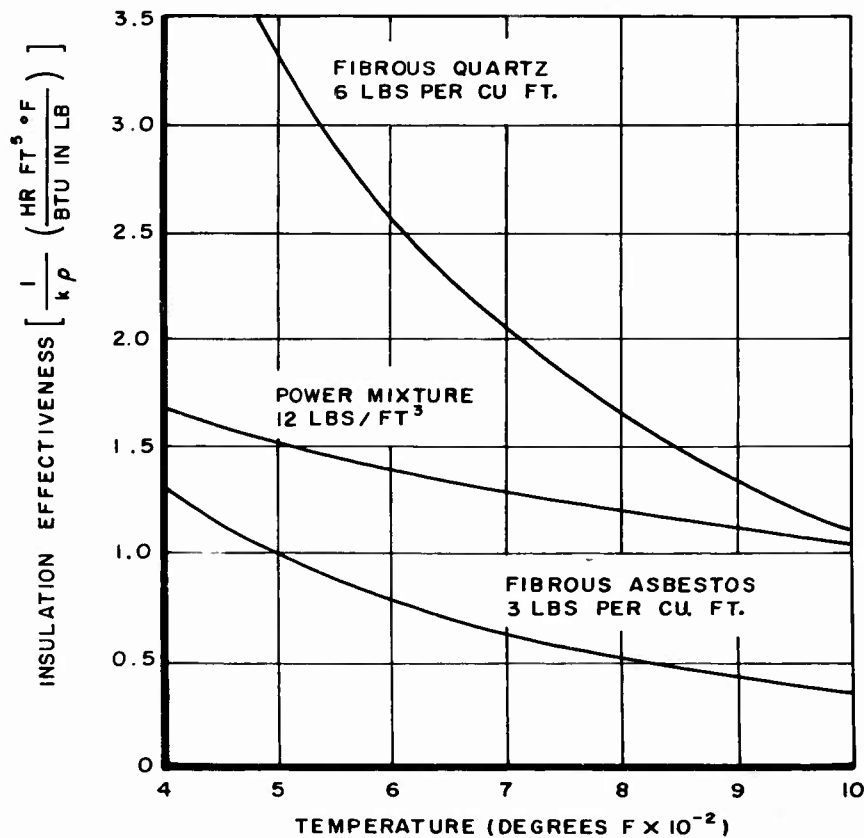


Figure 5. Comparison of the Effectiveness of Insulations

for a more accurate comparison. These effects are included simply by a weighted adjustment of both the thermal conductivity and density.

Additional important characteristics of insulating materials are the following:

- a. Adequate compressive strength to insure retention and prevent settling;
- b. Softening point above the maximum operating temperature to prevent a loss of compressive strength and particle shape;
- c. Minimum contact area between individual particles to reduce solid conduction;
- d. Absence of sintering at operating temperatures to prevent degradation of material properties;
- e. Ability to attenuate radiant heat transfer by particle orientation and emissive characteristics, which cause sufficient radiation blockage;
- f. Particle size distribution and construction to minimize compaction under vibrational or gravitational forces;
- g. Low vapor pressure at maximum operating temperatures to preclude loss of the material;
- h. Particle size that is so small that the mean free path of the air molecules becomes larger than the particle spacing at moderate vacuums.

Powder materials and fibrous forms of insulation have physical characteristics that effectively reduce each of the contributing modes of heat transfer and satisfy the preceding requirements. Gas conduction, for example, is lowered by the orientation of fine particles to create pore sizes, which, at moderate vacuums, are smaller than the mean free path that the gas molecules travel before colliding. Constituent materials for the insulation are chosen on the basis of their ability to attenuate heat flow through the materials. Major constituents such as alumina, potassium titanate, and fibrous asbestos have excellent insulating qualities, which, when coupled with the many joint resistances at the points where particles contact one another, resist the flow of heat by solid conduction. Thermal radiation can be reduced by the addition of both radiation absorbing particles (such as zirconia, carbon, and silicon nitride) and radiation reflecting or scattering particles (such as aluminum, platinum, or tantalum flakes). The effect of temperature and altitude on the thermal conductivity of an insulation powder is shown in Figure 6.

Although fine powders permit a wider flexibility in selecting ingredients for specific temperature applications than do fibrous mats, problems that are associated with the packaging of these materials tend to overshadow any relative advantages. Specific problems include the following:

- a. Compaction and settling of constituents under vibratory loads,
- b. Excessive resistance to air flow during boost venting,
- c. Development of filter configurations to retain fine particles,
- d. Insulation package buckling due to thermal stress.

The more common fibrous insulations contain either aluminum silicate or silica fibers, which are useful to 2000°F. These are commercially available in various forms, namely, bulk, batt, blanket, cloth, molded shapes, and paper. The diameters of the fibers are in the range from 0.00003 to 0.0004 inch and the densities range from 3 to 26 pounds per square foot. The thermal conductivity for one of the fibrous quartz insulations is shown in Figure 7 as a function of temperature and altitude.

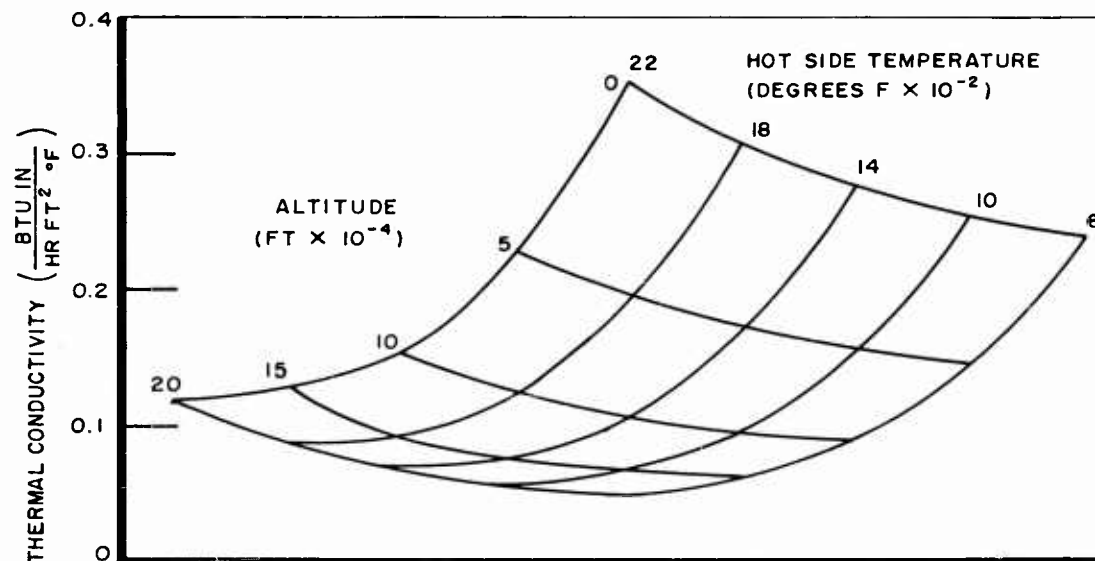


Figure 6. Effect of Temperature and Altitude on Thermal Conductivity of Powdered Mixture

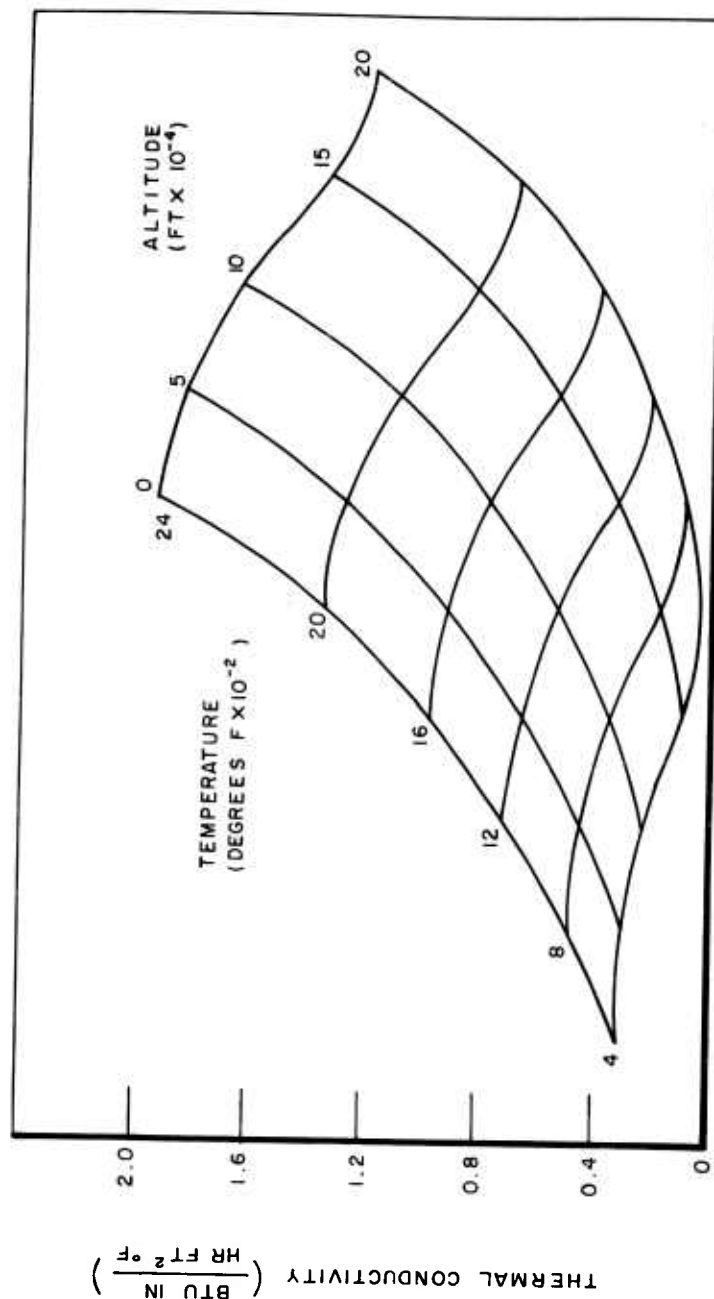


Figure 7. Effect of Temperature and Altitude on Thermal Conductivity of Fibrous Quartz Insulation

1.2.3.3 Cooling Systems

Three cooling system concepts appear feasible for use in removing the aerodynamic heat that is transmitted through the insulation. These systems are defined in the succeeding paragraphs.

1.2.3.3.1 Open-Cycle Systems

The coolant of the open-cycle systems (a 99-percent water, 1-percent gel mixture) is contained by a sponge material within passages that are attached to a thin aluminum sheet. The assembled panels are mounted to the pilot and payload compartments; they enclose areas containing equipment that cannot tolerate exposure to the hot surface panels. When transmitted through the protective insulation, heat is absorbed by vaporizing the coolant. The steam generated is transported to an overboard dump where it is exhausted through a control valve, which functions primarily to regulate the coolant temperature. Another advantage of a pressurized system is the elimination of coolant vaporization during orbit since the vapor pressure of the gel would exceed the pressure within the passages. A schematic of this system is shown in Figure 8.

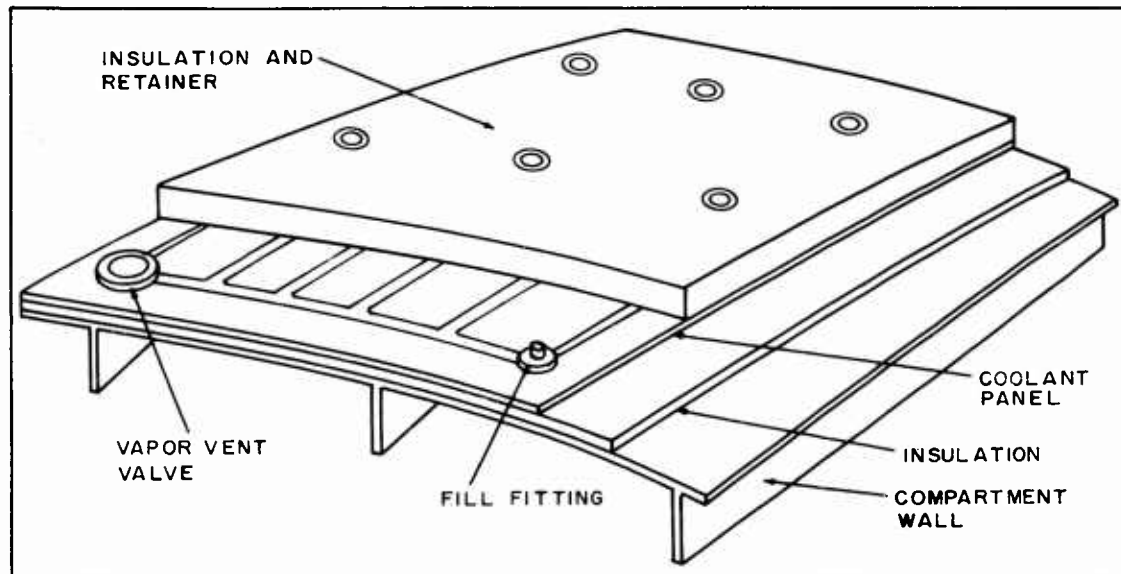


Figure 8. Open Coolant System Integrated with Protection System

1.2.3.3.2 Closed-Cycle Systems

The compartment shell or primary structure of the closed-cycle systems, constructed from an aluminum alloy, contains an integral tube circuit that forms a closed transport loop through which a coolant is circulated. During re-entry, heat transmitted through the insulation is absorbed by the circulating fluid and transferred in a remote heat exchanger to an expendable heat sink that uses its heat of vaporization in the process. The rate of flow of the expendable coolant through the heat exchanger regulates the temperature in the closed loop. Figure 9 shows a schematic of a closed-cycle system.

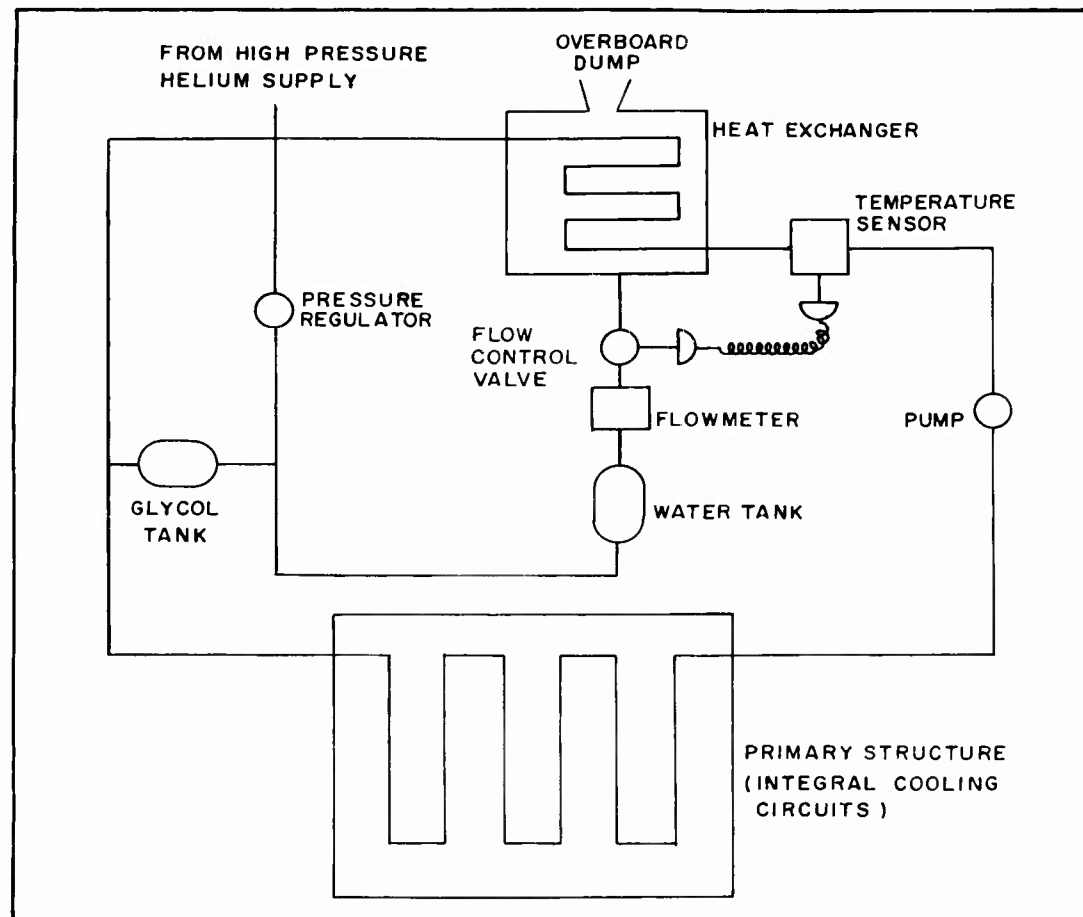


Figure 9. Closed-Cycle Cooling System

1.2.3.3.3 Combined Open-Cycle and Closed-Cycle Systems

The combined open-cycle and closed-cycle systems can be used advantageously to enhance the overall configuration of the system. In general, the closed-cycle systems can be adapted readily to large surface areas and thermally complex sections where local conductance values vary significantly. Areas where appreciable heat conduction results from structural framework that support hatches and windshields, and members that extend between the outer heat shields and the cooled inner structure or compartment walls are examples of this case. On the other hand, remote equipment and movable surfaces might be more effectively cooled by an open system to eliminate long or flexible connections, which would be more susceptible to fatigue.

1.3 System Optimization

In the final analysis, many design parameters might influence optimization procedures. These include allowable material temperatures, reliability considerations, volume limitations, flexibility and fabrication requirements of the system, systems that are readily accessible for servicing, and ease of adaptation to straightforward preflight checkout procedures. In addition, the thermal protection system must be able to survive environments that are associated with ground support as well as all phases of an orbital mission.

The integration of the components of thermal protection systems into a system of minimum weight can be performed after the type of insulation and cooling system have been selected. Essentially, the optimization process involves a tradeoff between insulation and coolant requirements to arrive at a combination of minimum weight. The following sections present the analytical techniques associated with this process.

2.0 SUMMARY

A thermal analysis and an analytical procedure of practical significance are presented and applied to establish meaningful design parameters for thermally protected glide vehicles that re-enter the atmosphere. With only slight modification, as dictated by particular boundary conditions, the techniques developed may be useful for obtaining solutions to a number of problems in the field of heat transfer where temperatures are functions of spatial and time coordinates.

The general differential heat conduction or diffusion equation, extended to include temperature-dependent thermal properties, was transformed into an approximate algebraic expression by employing the calculus of finite differences. The difference terms replacing the partial derivatives were the first approximations of the derivatives of polynomials, obtained from the Gregory-Newton and Stirling interpolation formulas, which represented the temperature at any point in the coordinate system.

Three realistic thermal protection systems were analyzed by the application of this general difference equation together with the appropriate boundary conditions. Three configurations were selected: an insulated and cooled structure, an insulated and cooled compartment combined with a "hot" structure, and an internally radiation-cooled structure. The boundary value problems in each case were reduced to expressions that enabled a prediction of the temperature history throughout the configuration, instantaneous rates of heat transfer to the cooling system, as well as the total coolant requirements that were based on the amount of heat absorbed during a typical descent trajectory.

The equations resulting from the analysis of the insulated and cooled structure were programmed on an IBM 7090 computer and numerical solutions were obtained. Temperature surveys for this case are presented in the results and the significance of optimizing the design is discussed.

3.0 THERMAL ANALYSIS

3.1 Problem Statement

Optimization procedures establish, by an orderly analytical process, the insulation and cooling requirements for thermal protection systems of minimum weight. These requirements are set forth by arranging the problem solutions in an expression for the total system weight as a function of insulation thickness and then by selecting the thickness, which corresponds to the minimum combined weights. The techniques involved in generating sufficient data to adequately define this function will be presented in this section.

The total weight of the system can be divided into two basic elements; the weight of the insulation that is assumed as an input to the problem and the coolant weight that is expended as a result of absorbing heat, which is conducted through the insulation from the surface of the vehicle. The latter quantity can be determined from the solution of the boundary value problem for conditions that define the time-dependent flow of heat through the insulation. One condition on U , which is a familiar parabolic partial differential equation, can be written in the form

$$U_t = G(U) \quad (1)$$

where the right-hand term is a second order elliptic partial differential operator with either one, two, or three independent space variables. Now if R is the region of the space variable or variables bounded by D , then the values of U in R can be determined for all $t > t_0$ after U in R is specified for $t = t_0$ and the values of U on D are known for all $t > t_0$.

Since obtaining solutions to this problem by analytical methods is not practical, an approximate numerical method can be used which replaces the partial differential equation by a partial difference equation. This method, described here, is the method of finite differences.

A grid network of regularly spaced straight lines, each parallel to one of the coordinate axis, is superimposed over the region R . In addition, a time increment, p , is introduced, which essentially represents a new dimension to the mesh. Now instead of solving a complex differential equation for all values in a continuous region, only approximate values of the solution must be obtained at the mesh points that are formed by the intersections of the network lines with each other and with the boundary of the region.

The method of obtaining the difference equation for each of the interior mesh points involves a transformation of the differential equation by replacing the partial derivatives with corresponding difference quotients. As a result, the problem can be formulated into a series of linear algebraic equations where each defines the conditions at a particular interior mesh point. In the case of the boundary value problems that are considered in the following sections, the solutions will involve a system of n linear algebraic equations with n unknowns, where n is the number of interior mesh points. All values pertaining to the boundary mesh points will be prescribed.

Since the initial values for U for the interior points are known for $t = t_0$, the solutions for the first set of n algebraic equations will apply for $t = t_0 + p$. Successive iterations will result in approximate numerical values of U for all mesh points in R when $t = t_0 + 1p, t_0 + 2p, \dots, t_0 + mp$ where m is the total number of time increments.

An explicit procedure for solving boundary value problems of the type containing Equation (1) is referred to as the forward difference method. Each value of U can be determined for any value of t if all values for U at the preceding t are known. Unfortunately, the forward difference method may be numerically unstable unless restrictive conditions are imposed upon a parameter for selecting p . As a result, the maximum value for p is limited; nevertheless, for most cases p must be chosen so small relative to the space size of the mesh that the computational labor involved prohibits solutions except where high-speed digital computers are available.

Once the values of U have been established for all mesh points within R and for $t = t_0 + 1p, t_0 + 2p, \dots, t_0 + mp$, then the rate of heat flow, q , across the boundary can be calculated. The total heat removed, Q , can be determined by a summation of q .

$$mp \sum_{i=1}^m q \quad (2)$$

From this term, the total weight of the coolant expended is determined by dividing by the enthalpy that is required to convert the liquid coolant to a gaseous state. This process can then be repeated for a number of initial insulation thicknesses to arrive at some relationship between this independent variable and the coolant requirements.

The introduction and availability of high-speed computational machines have made practical the solution of this problem by numerical methods. Unfortunately, considerable effort is involved in formulating the problem into computer language and preparing the machine program. As a result, the time lag between the start and solution of a problem might be significant but of less relative importance where a great many cases are to be solved. One valuable aid for expediting problem processing is an adaptation of a concise format for presenting solution procedures. Section 4 contains the problem statement in a recommended form that can be understood by engineers as well as programmers.

3.2 General Assumptions

Before presenting a solution to the time-dependent heat transfer or diffusion equation, certain assumptions suggested by the physical nature of the problem should be reviewed. These involve the selection of significant spatial variables, the manner of analytically expressing material properties, and general definitions pertaining to boundary conditions.

The first problem can be resolved by determining the relative significance of heat transfer with respect to the coordinate system shown in Figure 10. These coordinates pass through an element of insulation in an area of the vehicle where the orientation of isotherms results in maximum heat-flow rates. Since the thermal conductivity of the material is identical in all three directions, then only the ratio of temperature gradients in these directions need be compared.

Temperature variations with distance can be expected to reach a maximum in a section directly behind the leading edge of the wing where the value for $\frac{\Delta u(x) \Delta y}{\Delta u(y) \Delta x} = 50$, or only

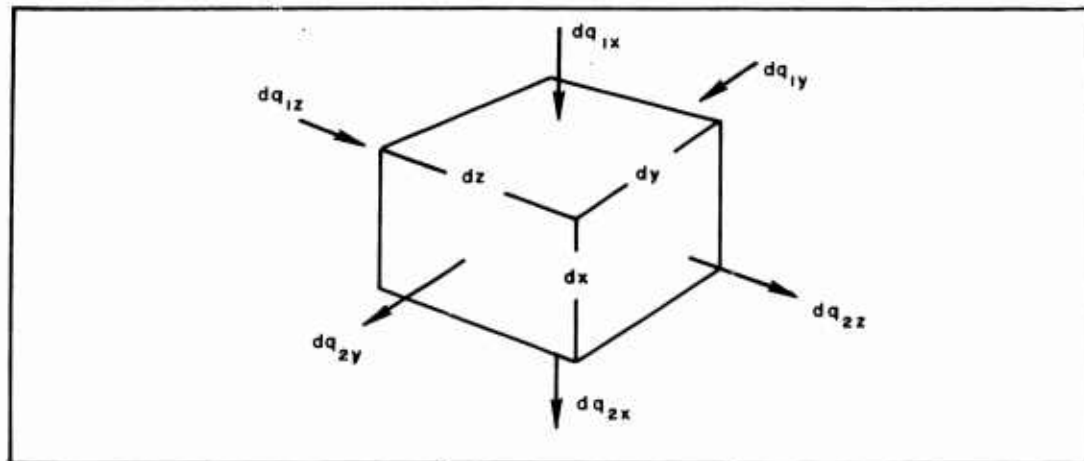


Figure 10. Element of Insulation

2 percent of the heat conducted depthwise through the structure will be transmitted in a chordwise direction. In Section 5, you will note that the total weight of the system will not be affected by this small percentage. The heat transferred in a spanwise direction is still smaller; in fact, it can be considered as nil.

Another section of the vehicle that was chosen for evaluating the merits of a three-dimensional heat-transfer analysis was the upper, rear portion of the fuselage where the surface temperatures during the period of maximum re-entry heating are relatively low. In this area, the value of the ratio $\frac{\Delta u(x) \Delta z}{\Delta u(z) \Delta x} = 200$, which indicates that the radial heat transfer far exceeds the circumferential conduction; hence, the latter could be neglected. The same approach can be applied to justify neglecting heat that is transferred axially along the fuselage.

Although we have shown that analytical solutions in the practical sense will be sufficiently accurate if, in general, only one-dimensional heat transfer is assumed, a numerical approximation of the diffusion equation in three spatial variables will be derived, since it provides a valuable tool for evaluating a number of engineering problems in heat transfer. The matter of transforming the derivation to a one-dimensional equation is accomplished merely by retaining only those terms applicable to the particular problem in question.

During re-entry, the temperatures of the external heat shields may be related in some manner to mission time. Although no exact mathematical formulation is possible for this function, it may best be approximated by a series of points; its neighborhood can be defined by linear interpolation. This assumption is not as wild as it may appear, since digital computer techniques can readily handle this procedure.

Another general assumption is associated with the mathematical treatment of material thermal properties; namely, surface emissivity, thermal conductivity, and specific heat.

The thermal emissivity of the material surfaces is assumed to be independent of temperature. This assumption is valid for two reasons. First, the emissivity of surfaces experiencing an appreciable temperature variation (in the order of 2000°F) during re-entry are essentially grey bodies. Secondly, all nongrey surfaces experience only a small change in temperature, usually less than 100°F.

Thermal conductivity and specific heat, on the other hand, are definitely functions of temperature. If these properties are inspected for a number of applicable materials, we see that thermal conductivity can be reasonably defined by a quadratic term while specific heat can be approximated by a linear function. This approximation results in expressions that are easily operated upon and, in addition, provides a convenient means of reducing the degree of the equations by setting coefficient equal to zero. For example, if the thermal conductivity is best represented by a linear function, a second degree polynomial can be replaced by a linear function merely by equating the coefficient of the squared term to zero. Assumptions concerning the selection of a value for the radiation form or geometry factor and a decision to omit air convection terms have been made in the interest of generalizing solutions. Since both of these terms are related to specific locations of vehicle surfaces, including particular design considerations would be necessary for all cross-sectional areas of a thermally protected airframe.

Many other significant assumptions could be discussed; however, they are more closely associated with particular boundary-value problems and would not apply in general. These assumptions will be reserved for Section 3.4 where particular boundary value problems will be analyzed.

3.3 Solution of the Diffusion Equation

The partial differential equation describing the time-dependent flow of heat through the thermal protection system is derived in this section; it is then transformed into an algebraic expression by the application of the calculus of finite differences. The final form is one that expresses the temperature of a finite element at time $t + 1$, in terms of the temperature of elements at time t . By successive application of this equation together with appropriate boundary and initial conditions, the temperature history through the insulation may be predicted; thereby, the rate of heat transfer from the n^{th} element is established. This latter quantity establishes the cooling weight of the system for any given amount of applied insulation.

Note Figure 10. The heat entering face 1x can be written as

$$dQ_x = dydz k(u) \frac{\partial u}{\partial x} dt \quad (3)$$

Now the heat leaving face 2x at $x + dx$ can be determined by letting

$$F(x, u) = k(u) \frac{\partial u}{\partial x} \quad (4)$$

By changing x to $x + dx$, $F(x, u)$ becomes $F(x + dx, u)$. This latter term may be expended by using a Taylor series, thus

$$F(x + dx, u) = F(x, u) + \frac{\partial F}{\partial x} dx = k(u) \frac{\partial u}{\partial x} + \frac{\partial}{\partial x} k(u) \frac{\partial u}{\partial x} dx \quad (5)$$

Therefore, the heat leaving face 2x can be expressed as

$$dQ_{x+dx} = -dydz \left[k(u) \frac{\partial u}{\partial x} + \frac{\partial}{\partial x} k(u) \frac{\partial u}{\partial x} dx \right] dt \quad (6)$$

This same approach would apply to adjacent y and z faces. That is

$$dQ_y = -dx dz k(u) \frac{\partial u}{\partial y} dt, \quad (7)$$

$$dQ_{y+dy} = -dx dy \left[k(u) \frac{\partial u}{\partial y} + \frac{\partial}{\partial y} k(u) \frac{\partial u}{\partial y} dy \right] dt \quad (8)$$

and

$$dQ_z = -dx dy k(u) \frac{\partial u}{\partial z} dt, \quad (9)$$

$$dQ_{z+dz} = -dx dy \left[k(u) \frac{\partial u}{\partial z} + \frac{\partial}{\partial z} k(u) \frac{\partial u}{\partial z} dz \right] dt \quad (10)$$

Since the net heat flow into an element must be equal to the heat stored within the element,

$$dQ_x + dQ_y + dQ_z = dQ_{x+dx} + dQ_{y+dy} + dQ_{z+dz} + \rho c_p dx dy dz \frac{du}{dt} dt. \quad (11)$$

When the dQ terms are replaced by the expressions Equations (3), (6), (7), (8), (9), and (10), all terms to the left in Equation (11) are cancelled by the first terms of the first three expressions on the right. The result is the following three-dimensional conduction equation:

$$\frac{\partial}{\partial x} k(u) \frac{\partial u}{\partial x} + \frac{\partial}{\partial y} k(u) \frac{\partial u}{\partial y} + \frac{\partial}{\partial z} k(u) \frac{\partial u}{\partial z} = \rho c_p \frac{\partial u}{\partial t}. \quad (12)$$

Equation (11) can be expanded to

$$\begin{aligned} \frac{\partial}{\partial x} k(u) \frac{\partial u}{\partial x} + k(u) \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial y} k(u) \frac{\partial u}{\partial y} + k(u) \frac{\partial^2 u}{\partial y^2} + \\ \frac{\partial}{\partial z} k(u) \frac{\partial u}{\partial z} + k(u) \frac{\partial^2 u}{\partial z^2} = \rho c_p \frac{\partial u}{\partial t} \end{aligned} \quad (13)$$

Equation (13) can now be transformed into an algebraic expression by substituting appropriate finite difference relationships for the partial derivatives. These approximations, derived from interpolation formulas, are listed as follows:

$$\frac{\partial u}{\partial x} = \frac{1}{2h} [u(i+1, x, t) - u(i-1, x, t)], \quad (14)$$

$$\left(\frac{\partial u}{\partial x} \right)^2 = \frac{1}{4h^2} [u(i+1, x, t) - u(i-1, x, t)]^2, \quad (15)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h^2} [u(i+1, x, t) - 2u(i, t) + u(i-1, x, t)], \quad (16)$$

$$\frac{\partial u}{\partial t} = \frac{1}{p} [u(i, t+1) - u(i, t)] \quad (17)$$

$$\frac{\partial u}{\partial x} = \frac{1}{h} [u(i+1, x, t) - u(i, t)] \quad (18)$$

Although these expressions have been written for the x direction only, those for the y and z spatial variables are similar. In addition, a square element was selected to eliminate area terms from the final equations. The derivation of these expressions is included in Appendix I.

Now the thermal conductivity of the material can be expressed as a function of temperature by the polynomial

$$k(u) = A_0 + A_1 u + A_2 u^2 \quad (19)$$

and the specific heat by

$$c_p(u) = B_0 + B_1 u \quad (20)$$

Taking the derivative of Equation (19) and dividing by ∂x gives

$$\frac{\partial k(u)}{\partial x} = (A_1 + 2A_2 u) \frac{\partial u}{\partial x} \quad (21)$$

Substituting Equation (21) into Equation (13) results in the expression

$$\begin{aligned} & A_1 + 2A_2 u \left(\frac{\partial u}{\partial x} \right)^2 + A_0 + A_1 u + A_2 u^2 \left(\frac{\partial^2 u}{\partial x^2} \right) + A_1 + \\ & 2A_2 u \left(\frac{\partial u}{\partial y} \right)^2 + A_0 + A_1 u + A_2 u^2 \left(\frac{\partial^2 u}{\partial y^2} \right) + A_1 + 2A_2 u \left(\frac{\partial u}{\partial z} \right)^2 + \\ & A_0 + A_1 u + A_2 u^2 \left(\frac{\partial^2 u}{\partial z^2} \right) = \rho (B_0 + B_1 u) \frac{\partial u}{\partial t} \end{aligned} \quad (22)$$

Replacing the derivatives with the appropriate finite difference relationships gives

$$\begin{aligned} & \frac{1}{4h^2} [A_1 + 2A_2 u(i, t)] \{ [u^2(i+1, x, t) - 2u(i+1, x, t)u(i-1, x, t) + u^2(i-1, x, t)] + \\ & [u^2(i+1, y, t) - 2u(i+1, y, t)u(i-1, y, t) + u^2(i-1, y, t)] + \\ & [u^2(i+1, z, t) - 2u(i+1, z, t)u(i-1, z, t) + u^2(i-1, z, t)] \} + \\ & \frac{1}{h^2} [A_0 + A_1 u(i, t) + A_2 u^2(i, t)] \{ [u(i+1, x, t) - 2u(i, t) + u(i-1, x, t)] + \\ & [u(i+1, y, t) - 2u(i, t) + u(i-1, y, t)] + [u(i+1, z, t) - 2u(i, t) + u(i-1, z, t)] \} = \\ & \frac{\rho}{p} [B_0 + B_1 u(i, t)] [u(i, t+1) - u(i, t)] \end{aligned} \quad (23)$$

Finally, solving for $u(i, t+1)$ gives

$$\begin{aligned}
 u(i, t+1) = & u(i, t) + \frac{P}{20 \rho h^2 [B_0 + B_1 u(i, t)]} \left\{ A_1 + 2 A_2 u(i, t) \right. \\
 & \left[u^2(i+1, x, t) - 2u(i+1, x, t) u(i-1, x, t) + u^2(i-1, x, t) + u^2(i+1, y, t) - \right. \\
 & 2u(i+1, y, t) u(i-1, y, t) + u^2(i-1, y, t) + u^2(i+1, z, t) - 2u(i+1, z, t) u(i-1, z, t) + \\
 & \left. u^2(i-1, z, t) \right] + 4 [A_0 + A_1 u(i, t) + A_2 u^2(i, t)] \\
 & \left[u(i+1, x, t) + u(i-1, x, t) + u(i+1, y, t) + u(i-1, y, t) + \right. \\
 & \left. u(i+1, z, t) + u(i-1, z, t) - 6u(i, t) \right] \left. \right\} \quad (24)
 \end{aligned}$$

For the case of one-dimensional heat flow, all terms associated with y and z directions will drop out and Equation (24) reduces to

$$\begin{aligned}
 u(i, t+1) = & u(i, t) + \frac{P}{20 \rho h^2 [B_0 + B_1 u(i, t)]} \\
 & \left\{ A_1 + 2 A_2 u(i, t) [u^2(i+1, t) - 2u(i+1, t) u(i-1, t) + u^2(i-1, t)] + \right. \\
 & \left. 4 [A_0 + A_1 u(i, t) + A_2 u^2(i, t)] [u(i+1, t) - 2u(i, t) + u(i-1, t)] \right\} \quad (25)
 \end{aligned}$$

This is an expression for the temperature of an element of insulation of length h and of unit area at time $t+1$, in terms of temperature-dependent thermal properties and the temperature of the same elements at time t .

3.4 Boundary Value Problems

A number of typical boundary value problems that were encountered during the formulation of optimization procedures for thermal protection systems are presented in this section. Each problem is associated with some definite combinations of thermal insulation and cooling system or thermal insulation without a cooling system, which are representative of schemes evolving from current technological developments. The following types will be analyzed in the following sections: insulated and cooled structure, insulated and cooled compartments, and insulated structure.

3.4.1 Insulated and Cooled Structure

A cross section through the wall of an insulated and cooled structure is shown in Figure 11. Let s represent the external surface of the radiation shields, o the interface between the heat shield and thermal insulation, r the inboard surface of the insulation,

c the cooled structure, and g the compartment atmosphere and insulation interface.

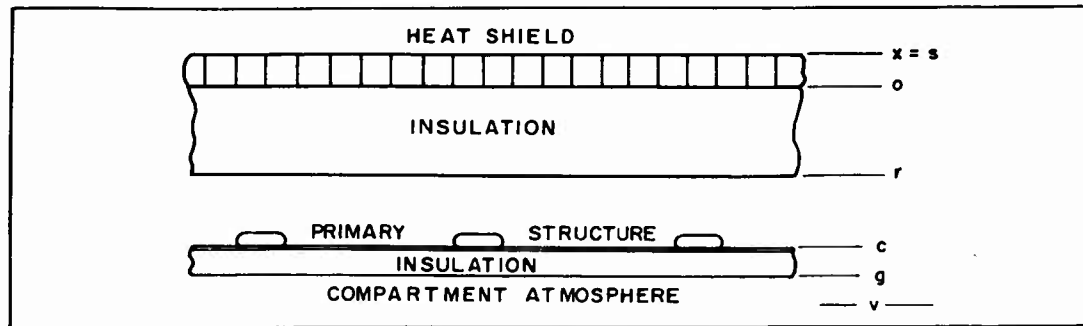


Figure 11. Insulated and Cooled Structure

The conditions of the boundary value problem are stated as follows:

Considering the outer layer of insulation (o to r) gives

$$U_t(x, t) = G(U) \quad 0 < x < r, t > 0, \quad (26)$$

$$U_x(x, t) = \frac{C_b}{k(u)} [u(s, t) - u(o, t)] \quad x = 0, t > 0, \quad (27)$$

$$U_x(x, t) = \frac{C_a + C_r}{k(u)} [u(r, t) - u(c)] \quad x = r, t > 0, \quad (28)$$

and

$$U(x, 0) = \text{constant} \quad 0 < x < r, t = 0. \quad (29)$$

A similar set of conditions applies to insulation contained by the cooled structural shell:

$$U_t(x, t) = G(u) \quad c < x < g, t > 0, \quad (30)$$

$$U_x(x, t) = \frac{C_a + C_r}{k(u)} [u(r, t) - u(c)] \quad x = c, t > 0, \quad (31)$$

$$U_x(x, t) = \frac{C_h}{k(u)} [u(g, t) - u(v)] \quad x = g, t > 0, \quad (32)$$

$$U(x, 0) = \text{constant} \quad c < x < g, t = 0. \quad (33)$$

The problem statement can now be completed by stating the remaining relationships that affect a solution thus

$$U(s, t) = F(t) \quad x = s, t \geq 0, \quad (34)$$

$$U(c, t) = \text{constant} \quad x = c, t \geq 0, \quad (35)$$

$$U(v, t) = \text{constant} \quad x = v, t \geq 0 \quad (36)$$

The preceding conditions can now be transformed into algebraic expressions by replacing the derivatives with the corresponding finite difference relationships listed in Section 3.3, Equations (14) through (18).

The transformation of the identical Equations (26) and (30) was discussed in Section 3.3. As a result, Equation (25) was derived. Similar operations on Equation (27) will now be performed:

$$k(u) \frac{\partial u}{\partial x} \bigg|_{x=0} = C_b [u(s, t) - u(0, t)] \quad (37)$$

where C_b is the conductance of the radiation heat shield. Referring to Figure 12,

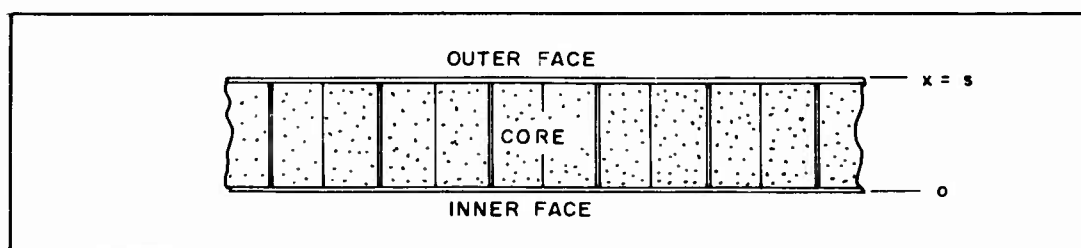


Figure 12. Heat Shield

$$C_b = C_d + C_a + C_r \quad (38)$$

states that the overall conductance of a honeycomb sandwich panel is the sum of the conductance through the metal core, C_d , and the conductances of the air space $C_a + C_r$. Now

$$C_d = \frac{k_d(u) S_d}{l_d}, \quad C_a = \frac{k_a(u) S_a}{l_d}, \quad C_r = C_r'(u) S_a \quad (39)$$

But C_b can be approximated by a quadratic

$$C_b = D_0 + D_1 u + D_2 u^2 \quad (40)$$

completing the transformation

$$\begin{aligned} \frac{1}{h} \left\{ A_0 + \frac{A_1}{2} [u(0, t+1) + u(0+1, t+1)] + \frac{A_2}{4} [u^2(0, t+1) + \right. \\ \left. 2u(0, t+1)u(0+1, t+1) + u^2(0+1, t+1)] \right\} [u(0, t+1) - u(0+1, t+1)] = \\ D_0 + \frac{D_1}{2} [u(s, t+1) + u(0, t+1)] + \frac{D_2}{4} [u^2(s, t+1) + \\ 2u(s, t+1)u(0, t+1) + u^2(0, t+1)] [u(s, t+1) - u(0, t+1)] \end{aligned} \quad (41)$$

Expanding, collecting, and transposing terms, Equation (41) becomes

$$\begin{aligned} & (A_2 + D_2 h) u^3(o, t+1) + [2A_1 + A_2 u(o+1, t+1) + 2hD_1 + hD_2 u(s, t+1)] u^2(o, t+1) + \\ & [4hD_0 - hD_2 u^2(s, t+1) - A_2 u^2(o+1, t+1) + 4A_0] u(o, t+1) - [4A_0 u(o+1, t+1) + \\ & 2A_1 u^2(o+1, t+1) + A_2 u^3(o+1, t+1) + 4hD_0 u(s, t+1) + 2hD_1 u^2(s, t+1) + \\ & hD_2 u^3(s, t+1)] = 0. \end{aligned} \quad (42)$$

The term $u(s, t+1) = F(t)$ is given and $u(o+1, t+1)$ can be determined from Equation (25); therefore, $u(o, t+1)$ can be found by solving Equation (42).

The transformation of Equation (28) is accomplished in a similar fashion

$$k(u) \frac{\partial t}{\partial x} \Big|_{x=r} = C_a + C_r [u(r, t) - u(c)] \quad (43)$$

where

$$C_a = \frac{k_a(u)}{l_a} = \frac{1}{l_a} (A_{a0} + A_{a1} u_m + A_{a2} u_m^2) \quad (44)$$

where k_a is the thermal conductivity of air.

In Equation (44),

$$u_m = \frac{u(r, t+1) + u(c)}{2} \quad (45)$$

Also,

$$C_r = \sigma F_e F_a \left\{ \frac{[459.69 + u(r, t+1)]^4 - [459.69 + u(c)]^4}{u(r, t+1) - u(c)} \right\} \quad (46)$$

Expanding Equation (46)

$$\begin{aligned} C_r = \frac{\sigma F_e F_a}{u(r, t+1) - u(c)} & [u^4(r, t+1) + 1838.8 u^3(r, t+1) + \\ & 1267.9 \times 10^3 u^2(r, t+1) + 3885.6 \times 10^5 u(r, t+1) - \\ & u^4(c) - 1838.8 u^3(c) - 1267.9 \times 10^3 u^2(c) - 3885.6 \times 10^5 u(c)] \end{aligned} \quad (47)$$

Substituting Equations (44), (45), and (47) into Equation (43) together with the finite difference expression for the partial derivative yields the following:

$$\begin{aligned}
& \frac{1}{h} \left\{ A_0 + A_1 \left[\frac{u(r, t+1) + u(c)}{2} \right] + A_2 \left[\frac{u(r, t+1) + u(c)}{2} \right]^2 \right\} \\
& \left[u(r, t+1) - u(c) \right] = \left\{ A_{a0} + A_{a1} \left[\frac{u(r, t+1) + u(c)}{2} \right] + \right. \\
& \left. A_{a2} \left[\frac{u(r, t+1) + u(c)}{2} \right]^2 \right\} \left[\frac{u(r, t+1) - u(c)}{l_a} \right] + \sigma F_e F_a u^4(r, t+1) + \\
& 1838.8 \sigma F_e F_a u^3(r, t+1) + 1267.9 \times 10^3 \sigma F_e F_a u^2(r, t+1) + \\
& 3885.6 \times 10^5 \sigma F_e F_a u(r, t+1) - \sigma F_e F_a u^4(c) - 1838.8 \sigma F_e F_a u^3(c) - \\
& 1267.9 \times 10^3 \sigma F_e F_a u^2(c) - 3885.6 \times 10^5 \sigma F_e F_a u(c)
\end{aligned} \tag{48}$$

Equation (48) reduces to

$$\begin{aligned}
& 4h\sigma F_e F_a u^4(r, t+1) + \left(\frac{hA_{a2}}{l_a} + 7355.2 h\sigma F_e F_a - A_2 \right) u^3(r, t+1) + \\
& \left[\frac{2hA_{a1}}{l_a} + \frac{hA_{a2}}{l_a} u(c) + 5071.6 \times 10^3 h\sigma F_e F_a - 2A_1 - A_2 u(c) \right] u^2(r, t+1) + \\
& \left[\frac{4hA_{a0}}{l_a} + A_2 u^2(c) + 15542 \times 10^5 h\sigma F_e F_a - 4A_0 - \frac{hA_{a2}}{l_a} u^2(c) \right] u(r, t+1) + \\
& \left[4A_0 u(c) + 2A_1 u^2(c) + A_2 u^3(c) - \frac{4hA_{a0}}{l_a} u(c) - \frac{2hA_{a1}}{l_a} u^2(c) - \right. \\
& \left. \frac{hA_{a2}}{l_a} u^3(c) - 4h\sigma F_e F_a u^4(c) - 7355.2 h\sigma F_e F_a u^3(c) - \right. \\
& \left. 5071.6 \times 10^3 h\sigma F_e F_a u^2(c) - 15542 \times 10^5 h\sigma F_e F_a u(c) \right] = 0
\end{aligned} \tag{49}$$

Again, this equation can be solved for $u(r, t+1)$, since all other terms are known or can be determined by previously established relationships:

We assumed by Equation (35) that the circulating coolant media maintains a constant structural temperature; hence, the heat capacity of these elements does not affect the temperature gradient. Furthermore, the thermal conductivity of common materials used in this area are very high as compared with those of thermal insulations and, when coupled with typically thin sections, the temperature drop across the member is negligible.

Continuing with the solution, the temperature of internal elements on the inboard side of the structure can be determined by Equation (30). Again, the transformed relationship is identical to Equation (25).

Now Equation (31) states that the heat transferred from r to c is equal to the heat absorbed by the first element of insulation adjacent to c . A solution in this case was previously obtained (Equation (49)), since the temperature of c is constant; therefore, $u(c)$ is independent of conditions beyond that station.

The relationship remaining to be solved is Equation (32) thus

$$k(u) \frac{\partial u}{\partial x} \bigg|_{x=g} = C_h [u(g, t) - u(v)] \quad (50)$$

Again, the partial derivative is transformed to the difference relationship

$$\begin{aligned} \frac{1}{h} \left\{ A_0 + \frac{A_1}{2} [u(g-1, t+1) + u(g, t+1)] + \frac{A_2}{4} [u^2(g-1, t+1) + \right. \\ \left. 2u(g-1, t+1)u(g, t+1) + u^2(g, t+1)] \right\} [u(g-1, t+1) - u(g, t+1)] = \\ C_h [u(g, t+1) - u(v)] \end{aligned} \quad (51)$$

Expanding, grouping terms, and transposing, Equation (51) becomes

$$\begin{aligned} A_2 u^3(g, t+1) + [2A_1 - A_2 u(g-1, t+1)] u^2(g, t+1) + [4A_0 - A_2 u^2(g-1, t+1) + 4hC_h] u(g, t+1) - \\ [4A_0 u(g-1, t+1) + 2A_1 u^2(g-1, t+1) + A_2 u^3(g-1, t+1) + 4hC_h u(v)] = 0 \end{aligned} \quad (52)$$

The solution of Equation (52) for $u(g, t+1)$ completes the process of determining the temperature distribution through the thermal protection system at time $t+1$. A repetition of the process results in the temperature distribution at time $t+2$. By this technique, the temperature distribution for all t 's may be established.

An internal cooling system maintains $u(v)$ constant by circulating the compartment atmosphere across the face of the insulation. A similar condition existed at c where a fluid is circulated through passages that are attached to the structure. Determining the weights of coolant in each case requires that the rates of heat removal at c and v must first be calculated.

At the structure, c ,

$$q(c, t+1) = C_a + C_r [u(r, t+1) - u(c)] \quad (53)$$

since $u(c)$ is a constant, Equation (53) simply states that the rate of heat transfer to the coolant at time $t+1$ equals the rate of heat transfer across the air gap at time $t+1$. The total weight of coolant expended during re-entry is then

$$W_c = \frac{mp}{h_{fg}} \sum_1^m q(c, t). \quad (54)$$

Now the rate of heat transfer to the compartment

$$q(v, t+1) = C_h [u(g, t+1) - u(v)]$$

and the weight of expended coolant

$$W_v = \frac{mp}{h_{fg}} \sum q(v, t). \quad (55)$$

The amount of insulation in each case, which results in a minimum combined insulation and coolant weight, can be determined by repeating the solution for several insulation thicknesses; then the combined weights can be compared with the insulation thickness.

3.4.2 Insulated and Cooled Compartment

A typical section of this construction, consisting of an insulated radiation shield exposed to the airstream and an insulated cooling system attached to the compartment wall, is shown in Figure 13. The various surfaces and interfaces are identified as follows:

- a. At $x = s$, the exterior surface of the radiation shield is supported from a stiffened panel, f , by refractory metal members.
- b. At $x = 0$, a retaining sheet confines thermal insulation in an area that is adjacent to the cooling system, r . This assembly together with another layer of insulation is supported from the compartment wall, c .
- c. The compartment atmosphere is located at $x = v$.

The boundary value problems associated with this case can now be specified.

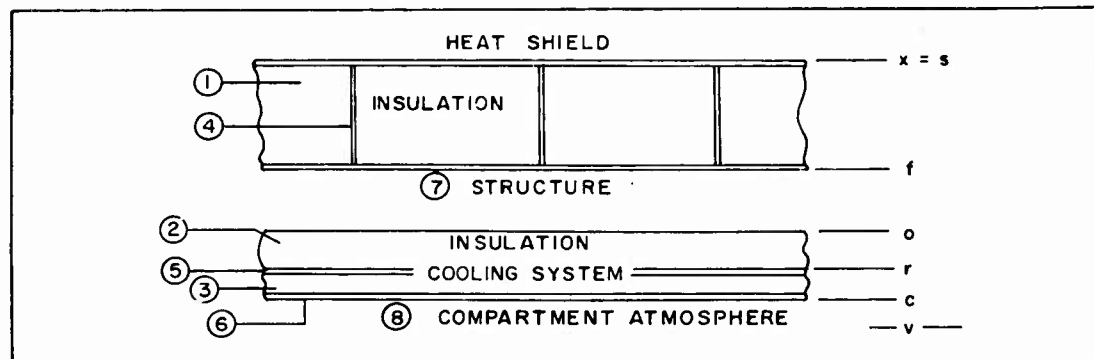


Figure 13. Hot Structure - Insulated and Cooled Compartment

Conditions for the insulated radiation shield are as follows:

$$U_t(x, t) = G(u) \quad s < x < f, t > 0, \quad (56)$$

$$U(s, t) = F_1(t) \quad x = s, \quad t \geq 0, \quad (57)$$

$$U_x(x, t) = \frac{1}{k(u)} \{ (C_a + C_r) [u(f, t) - u(0, t)] - C_s [u(s, t) - u(f, t)] \} \quad x = f, \quad t > 0, \quad (58)$$

$$U(x, 0) = \text{constant} \quad s < x < f, t = 0. \quad (59)$$

For the insulation between $x=0$ and $x=r$,

$$U_t(x, t) = G(u) \quad 0 < x < r, t > 0, \quad (60)$$

$$U_x(x, t) = \frac{1}{k(u)} \{ (C_0 + C_r) [u(f, t) - u(0, t)] \} \quad x = 0, t > 0, \quad (61)$$

$$U_x(x, t) = \frac{1}{k(u)} [q_s(r) + q_d(r) + q_0(r)] \quad x = r, t > 0, \quad (62)$$

$$U(x, 0) = \text{constant} \quad 0 < x < r, t = 0. \quad (63)$$

Finally, for the remaining section of insulation between $x=r$ and $x=c$

$$U_t(x, t) = G(u) \quad r < x < c, t > 0, \quad (64)$$

$$U_x(x, t) = \frac{1}{k(u)} [q_i(r) - (q_s(r) + q_d(r))] \quad x = r, t > 0, \quad (65)$$

$$U_x(x, t) = \frac{1}{k(u)} [q_s(c) + C_h(u(c, t) - u(v))] \quad x = c, t > 0, \quad (66)$$

$$U(x, 0) = \text{constant} \quad r < x < c, t = 0 \quad (67)$$

Relationships, in addition to those just listed, which will be required for a solution, are as follows:

$$U(r, t) = F_2(t) \quad x = r, t \geq 0, \quad (68)$$

$$h_{fg}(r) = F_3(t) \quad x = r, t \geq 0, \quad (69)$$

$$U(v, t) = \text{constant} \quad x = v, t \geq 0. \quad (70)$$

Following the procedure applied in Section 3.4.1, the partial derivatives of the boundary condition are transformed to finite difference expressions. Equations (56), (60), and (64) reduce, as before, to Equation (25).

Since the temperatures of $x=s$ for all t 's are given, and the temperature of internal elements of insulation for the first-time step, $t+1$, can be determined from the solution of Equations (56) and (60), then the next operation should produce the temperatures at $x=f$ and $x=0$ for this same time step. Now the rate of heat transfer from the insulation element at $x=f$ must be equal to the rate of heat transfer to the insulation element at $x=0$ if the material in the region $f \leq x \leq 0$ has zero heat capacity. This being the case for air, the two applicable Equations (58) and (61) at this location can be written as follows:

$$k(u) \frac{\partial u}{\partial x} \bigg|_{x=f} + C_s [u(s, t) - u(f, t)] = (C_0 + C_r) [u(f, t) - u(0, t)], \quad (71)$$

$$k(u) \frac{\partial u}{\partial x} \bigg|_{x=0} = (C_a + C_r) [u(f, t) - u(o, t)] \quad (72)$$

By solving these two expressions simultaneously, $u(f, t+1)$ and $u(o, t+1)$ may be determined. To simplify the algebra, we will formulate coefficients from thermal conductance terms.

First, the thermal conductivity of the material used for supports between $x=s$ and $x=f$

$$k_s(u) = A_{s0} + A_{s1} u_m + A_{s2} u_m^2 \quad (73)$$

$$u_m = \frac{u(s, t+1) + u(f, t)}{2} \quad (74)$$

Further, the thermal conductance for the support

$$C_s = \frac{k_s(u) S_s}{l_s} \quad (75)$$

Combining Equations (73) through (75) results in the following expression for C_s :

$$C_s = \frac{S_s}{4l_s} \left[4A_{s0} + 2A_{s1} u(s, t+1) + 2A_{s1} u(f, t) + A_{s2} u^2(s, t+1) + 2A_{s2} u(s, t+1) u(f, t) + A_{s2} u^2(f, t) \right] \quad (76)$$

This represents the expression for the first coefficient $C_1 = C_s$.

Now the thermal conductance for the element of insulation at $x=f$ is derived by combining the equation

$$k(u) = A_0 + A_1 u_m + A_2 u_m^2 \quad (77)$$

where

$$u_m = \frac{u(f-1, t+1) + u(f, t)}{2} \quad (78)$$

and h to obtain

$$C_i = \frac{l}{h} \left[A_0 + \frac{A_1}{2} u(f-1, t+1) + \frac{A_1}{2} u(f, t) + \frac{A_2}{4} u^2(f-1, t+1) + \frac{A_2}{2} u(f-1, t+1) u(f, t) + \frac{A_2}{4} u^2(f, t) \right] \quad (79)$$

For this case, let the second coefficient

$$C_2 = hC_i \quad (80)$$

The third coefficient is equated to the thermal conductance of the air space established by Equation (44), in which U_m is now

$$u_m = \frac{u(f, t) + u(o, t)}{2} \quad (81)$$

Therefore,

$$C_3 = C_3 = \frac{1}{4\ell_a} \left[4A_{a0} + 2A_{a1} u(f, t) + 2A_{a1} u(o, t) + A_{a2} u^2(f, t) + 2A_{a2} u(f, t) u(o, t) + A_{a2} u^2(o, t) \right] \quad (82)$$

Considering the relationship for the equivalent thermal conductance for air-gap radiation, an expression similar to Equation (47) is obtained. Rewriting Equation (47) and inserting the proper subscripts gives

$$C_r = \frac{C_4}{u(f, t+1) - u(o, t+1)} = \frac{1}{u(f, t+1) - u(o, t+1)} \left\{ \sigma F_e F_a \left[u^4(f, t) + 1838.8 u^3(f, t) + 1267.9 \times 10^3 u^2(f, t) + 3885.6 \times 10^5 u(f, t) - u^4(o, t) - 1838.8 u^3(o, t) - 1267.9 \times 10^3 u^2(o, t) - 3885.6 \times 10^5 u(o, t) \right] \right\} \quad (83)$$

Equation (83) shows that C_4 is equal to the term enclosed by the brackets.

The final coefficient is derived from the thermal conductance term for the element of insulation at $x = 0$ by a manner similar to that utilized in establishing Equations (77) through (80) where U_m is now

$$u_m = \frac{u(o, t) + u(o+1, t+1)}{2} \quad (84)$$

Now

$$\bar{C}_i = \frac{C_5}{h} = \frac{1}{h} \left[A_0 + \frac{A_1}{2} u(o, t) + \frac{A_1}{2} u(o+1, t+1) + \frac{A_2}{4} u^2(o, t) + \frac{A_2}{2} u(o, t) u(o+1, t+1) + \frac{A_2}{4} u^2(o+1, t+1) \right] \quad (85)$$

Recalling the simultaneous Equations (71) and (72), the partial derivatives may be replaced by finite terms and substituting the coefficients for the conductance and thermal conductivity results in the following:

$$\frac{C_2}{h} [u(f-1, t+1) - u(f, t+1)] + C_1 [u(s, t+1) - u(f, t+1)] = C_3 [u(f, t+1) - u(o, t+1)] + C_4 \quad (86)$$

$$C_3 [u(f, t+1) - u(o, t+1)] + C_4 = \frac{C_5}{h} [u(o, t+1) - u(o+1, t+1)] \quad (87)$$

Rearranging Equations (85) and (86) by transposing terms gives

$$\left(C_1 + C_3 + \frac{C_2}{h}\right) [u(f, t+1)] - C_3 u(o, t+1) = \frac{C_2}{h} u(f-1, t+1) + C_1 u(s, t+1) - C_4, \quad (88)$$

$$C_3 u(f, t+1) - \left(C_3 + \frac{C_5}{h}\right) u(o, t+1) = -C_4 - \frac{C_5}{h} u(o+1, t). \quad (89)$$

The temperature $u(f, t+1)$ and $u(o, t+1)$ can now be determined from Equations (88) and (89).

The next step involves determining the temperature of internal elements of insulation for $r < x < c$ at time $t+1$. Following the procedure employed for each of the preceding insulation thicknesses, these temperatures are determined from the expanded form of Equation (64).

When conditions at $x=r$ are explored, several possibilities may be encountered. These involve a knowledge of the rates of heat transfer about and at the junction r as specified by boundary conditions in Equations (62) and (65). Recalling that $x=r$ is the location of the cooling system implies that a heat sink exists at this point and the rate of heat dissipated by the coolant is denoted by q_d . Furthermore, the materials used in the construction of the cooling system can store heat at a rate equal to q_s . Now the heat involved in both mechanisms must be the balance of that transferred to the junction through the insulation at the rate q_i and the rate of heat transfer from the junction q_o so that the expressions for each can be written and acted upon accordingly, thus

$$q_i = k(u) \frac{\partial u}{\partial x} \Big|_{x \rightarrow -r} = \frac{1}{h} \left\{ A_0 + \frac{A_1}{2} [u(r-1, t+1) + u(r, t+1)] + \frac{A_2}{4} [u^2(r-1, t+1) + 2u(r-1, t+1)u(r, t+1) + u^2(r, t+1)] \right\} [u(r-1, t+1) - u(r, t+1)] \quad (90)$$

or

$$q_i = \frac{1}{4h} \left\{ 4A_0 u(r-1, t+1) - 4A_0 u(r, t+1) + 2A_1 u^2(r-1, t+1) - 2A_1 u^2(r, t+1) + A_2 u^3(r-1, t+1) + A_2 u^2(r-1, t+1)u(r, t+1) - A_2 u(r-1, t+1)u^2(r, t+1) - A_2 u^2(r, t+1) \right\}. \quad (91)$$

In a similar manner

$$q_o = k(u) \frac{\partial u}{\partial x} \Big|_{x \rightarrow +r} = \frac{1}{4h} \left\{ 4A_0 u(r, t+1) - 4A_0 u(r+1, t+1) + 2A_1 u^2(r, t+1) - 2A_1 u^2(r+1, t+1) + A_2 u^3(r, t+1) + A_2 u^2(r, t+1)u(r+1, t+1) - A_2 u(r, t+1)u^2(r+1, t+1) - A_2 u^3(r+1, t+1) \right\} \quad (92)$$

The heat stored

$$q_s = \rho_r c_{p_r} \ell_r \left. \frac{\partial u}{\partial t} \right|_{t+1} = \frac{5}{p} \left(\rho_1 c_{p_1} \ell_1 s_1 + \rho_2 c_{p_2} \ell_2 s_2 + \rho_3 c_{p_3} \ell_3 s_3 \right) [u(r, t+1) - u(r, t)] \quad (93)$$

The subscripts of the ρ , c_p , ℓ , and s terms refer to particular components of the cooling system.

Now the three conditions that apply at the junction $x = r$ are as follows:

$$a. q_i = q_o + q_s, \text{ then } q_d = 0 \quad (94)$$

$$b. q_i > q_o + q_s, \text{ then } q_i - (q_o + q_s) = q_d \quad (95)$$

c. $q_i < q_o + q_s$ implies that $u(r, t+1)$, although given, is incorrect since it leads to an "impossibility equation." The solution is to determine a new $u(r, t+1)$ that will satisfy the equality $q_i = q_o + q_s$. One must understand that $q_d = 0$ for this condition.

Combining Equations (91), (92), and (93) and solving for $u(r, t+1)$ gives

$$\begin{aligned} & \frac{A_2}{h} u^3(r, t+1) + \left\{ \frac{A_2}{4h} [u(r-1, t+1) + u(r+1, t+1)] \right\} u^2(r, t+1) + \\ & \left\{ \left[\frac{5}{p} (\rho_1 c_{p_1} \ell_1 s_1 + \rho_2 c_{p_2} \ell_2 s_2 + \rho_3 c_{p_3} \ell_3 s_3) \right] + \frac{2A_o}{h} + \frac{A_1}{h} - \frac{A_2}{4h} u^2(r-1, t+1) - \right. \\ & \left. \frac{A_2}{4h} u^2(r+1, t+1) \right\} u(r, t+1) = \frac{A_o}{h} u(r-1, t+1) + \frac{A_1}{2h} u^2(r-1, t+1) + \\ & \frac{A_2}{4h} u^3(r-1, t+1) + \frac{A_o}{h} u(r+1, t+1) + \frac{A_1}{2h} u(r+1, t+1) + \\ & \frac{A_2}{4h} u^3(r+1, t+1) + \left[\frac{5}{p} (\rho_1 c_{p_1} \ell_1 s_1 + \rho_2 c_{p_2} \ell_2 s_2 + \rho_3 c_{p_3} \ell_3 s_3) \right] u(r, t) \quad (96) \end{aligned}$$

The only temperature remaining as an unknown is the temperature at $x = c$. The boundary condition for this location is defined by Equation (66) or in slightly different terminology

$$k(u) \left. \frac{\partial u}{\partial x} \right|_{x=c} = \rho_c c_{p_c} \ell_c \left. \frac{\partial u}{\partial t} \right|_{t+1} + c_h [u(c, t+1) - u(v)] \quad (97)$$

Transforming the derivatives to finite difference terms and substituting in the expression for

$$k(u) = A_o + A_1 u_m + A_2 u_m^2 \quad (98)$$

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where

$$u_m = \frac{u(c-l, t+1) + u(c, t)}{2} \quad (99)$$

results in the following relationship

$$\begin{aligned} \frac{1}{h} \left[A_0 + \frac{A_1}{2} u(c-l, t+1) + \frac{A_1}{2} u(c, t) + \frac{A_2}{4} u^2(c-l, t+1) + \right. \\ \left. \frac{A_2}{2} u(c-l, t+1) u(c, t) + \frac{A_2}{4} u^2(c, t) \right] [u(c-l, t+1) - u(c, t+1)] = \\ \frac{5\rho_c c_{p_c} \ell_c}{\rho} [u(c, t+1) - u(c, t)] + C_h [u(c, t+1) - u(v)] \end{aligned} \quad (100)$$

To simplify the algebra, let

$$C_1 = \frac{1}{h} \left[A_0 + \frac{A_1}{2} u(c-l, t+1) + \frac{A_1}{2} u(c, t) + \frac{A_2}{4} u^2(c-l, t+1) + \right. \\ \left. \frac{A_2}{2} u(c-l, t+1) u(c, t) + \frac{A_2}{4} u^2(c, t) \right] \quad (101)$$

and

$$C_2 = \frac{5\rho_c c_{p_c} \ell_c}{\rho} . \quad (102)$$

Then

$$u(c, t+1) = \frac{1}{C_1 + C_2 + C_h} [C_1 u(c-l, t+1) + C_2 u(c, t) + C_h u(v)] \quad (103)$$

For this case, the weight of the coolant expended during re-entry is determined by summing up the q_d as determined by Equation (95).

$$W_r = m p \sum_1^m \frac{q(r, t)}{h_{fg_r}} \quad (104)$$

The heat-transfer rate to the compartment is

$$q(v, t) = C_h [u(c, t) - u(v)] \quad (105)$$

and the weight of the coolant required at this location is

$$W_v = m p \sum_1^m \frac{q(v, t)}{h_{fg_v}} . \quad (106)$$

The combination of insulation and coolant, which results in a minimum system weight, is determined as described in Section 3.4.1.

3.4.3 Insulated Structure

In some areas of re-entry glide vehicles such as wing surfaces where aerodynamic heating is much more intense on the lower surface than on the upper and the hotter sections internally view the colder ones, heat may be dissipated through the structure by radiation and gas conduction. This principle is illustrated in Figure 14.

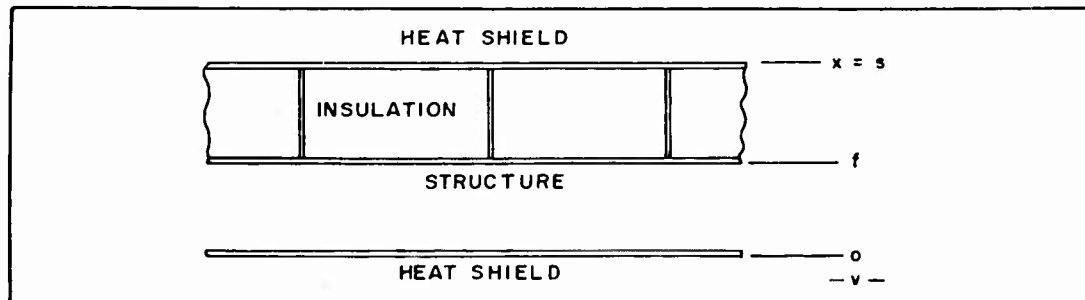


Figure 14. Hot Structure - Insulated

At $x = s$, the bottom surface of the wing, exposed to the air stream, is insulated to limit the temperature of the internal, load-carrying structure at a level where the stiffness is not significantly affected. Heat absorbed by the surface at $x = s$ is conducted through the insulation and the heat shield support to the surface at $x = f$. From this surface, it is transferred by radiation and gas conduction to the upper surface at $x = 0$, which is at a lower temperature because of the small heat input from the boundary layer. The problem in this case involves the application of sufficient insulation to obtain the desired temperature at $x = f$.

The conditions for the boundary value problem may be stated as follows:

In the region of the insulation

$$U_t(x, t) = G(u) \quad s < x < f, \quad t > 0, \quad (107)$$

$$U_x(x, t) = \frac{1}{k(u)} (q_g - q_r) \quad x = s, \quad t > 0, \quad (108)$$

where the heat generated at the lower surface

$$q_g = F_1(t) \quad x = s, \quad t > 0, \quad (109)$$

$$U_x(x, t) = \frac{1}{k(u)} \left\{ (C_d + C_r) [u(f, t+1) - u(0, t+1)] - C_s [u(s, t+1) - u(f, t+1)] \right\} \quad x = f, \quad t > 0, \quad (110)$$

$$U(x, 0) = \text{constant} \quad s \leq x \leq 0, \quad t = 0. \quad (111)$$

The equation that applies at $x = 0$ is

$$(C_a + C_r) [u(f, t+1) - u(0, t+1)] = q_r - \bar{q}_g \quad x = 0, t > 0. \quad (112)$$

Here \bar{q}_g is the heat generated at the upper surface where

$$\bar{q}_g = F_2(t) \quad x = 0, t > 0. \quad (113)$$

After first assuming some insulation thickness, the temperatures within the insulation for time $t + 1$ are calculated after first transforming Equation (107) to the form in Equation (25).

Now the temperature at $x = s$ for time $t + 1$ can be determined from Equation (108) where

$$k(u) \left. \frac{\partial u}{\partial x} \right|_{x=s} = q_g - \bar{q}_r \quad (114)$$

The term \bar{q}_r is that heat radiated to space from the surface of the heat shield.

Neglecting the temperature of heat sink in space gives

$$\bar{q}_r = \sigma F_e F_a [u^4(s, t+1) + 1838.8 u^3(s, t+1) + 1267.9 \times 10^3 u^2(s, t+1) + 3885.6 \times 10^5 u(s, t+1) + 4465.4 \times 10^7] \quad (115)$$

and

$$k(u) = A_0 + A_1 u_m + A_2 u_m^2 \quad (116)$$

where

$$u_m = \frac{u(s, t+1) + u(s+1, t+1)}{2}. \quad (117)$$

Equation (108) can be written as

$$\begin{aligned} & \frac{1}{h} \left\{ A_0 + \frac{A_1}{2} [u(s, t+1) + u(s+1, t+1)] + \frac{A_2}{4} [u(s, t+1) + u(s+1, t+1)]^2 \right\} \\ & [u(s, t+1) - u(s+1, t+1)] = q_g - \sigma F_e F_a [u^4(s, t+1) + 1838.8 u^3(s, t+1) + \\ & 1267.9 \times 10^3 u^2(s, t+1) + 3885.6 \times 10^5 u(s, t+1) + 4465.4 \times 10^7] \end{aligned} \quad (118)$$

Rearranging terms

$$\begin{aligned} & \sigma F_{\epsilon} F_0 u^4(s, t+1) + \left[\frac{A_2}{4h} + 1838.8 \sigma F_{\epsilon} F_0 \right] u^3(s, t+1) + \left[\frac{A_1}{2h} + \frac{A_2}{2h} u(s+1, t+1) - \right. \\ & \left. \frac{A_2}{4h} u(s+1, t+1) + 1267.9 \times 10^3 \sigma F_{\epsilon} F_0 \right] u^2(s, t+1) + \left[\frac{A_0}{h} + \frac{A_2}{4h} u^2(s+1, t+1) - \right. \\ & \left. \frac{A_2}{2h} u^2(s+1, t+1) + 3885.6 \times 10^5 \sigma F_{\epsilon} F_0 \right] u(s, t+1) = q_g + \frac{A_0}{h} u(s+1, t+1) + \\ & \frac{A_1}{2h} u^2(s+1, t+1) + \frac{A_2}{4h} u^3(s+1, t+1) - 4465.4 \times 10^7 \end{aligned} \quad (119)$$

Equation (119) can now be solved for $u(s, t+1)$.

Recognizing that Equation (110) is identical to Equation (58) then

$$\left(C_1 + C_3 + \frac{C_2}{h} \right) u(f, t+1) - C_3 u(o, t+1) = \frac{C_2}{h} u(f-1, t+1) + C_1 u(s, t+1) - C_4 \quad (120)$$

where

C_1 is given by Equation (76)

C_2 is h times Equation (79)

C_3 is given by Equation (82) and

C_4 is specified by Equation (83).

Now Equation (112), by the same process, can be written as

$$C_3 u(f, t+1) - C_3 u(o, t+1) = C_5 - \bar{q}_g - C_4 \quad (121)$$

where

$$\begin{aligned} C_5 = & \left[u(o, t+1) - u(v, t+1) \right] \bar{C}_r = \sigma F_{\epsilon} F_0 \left[u^4(o, t) + 1838.8 u^3(o, t) + \right. \\ & \left. 1267.9 \times 10^3 u^2(o, t) + 3885.6 \times 10^5 u(o, t) + 4465.4 \times 10^7 \right]. \end{aligned} \quad (122)$$

Equations (120) and (121) are solved simultaneously for $u(f, t+1)$ and $u(o, t+1)$.

If q_g and \bar{q}_g are known for all t , then $u(f)$ can be determined for all t . The process is repeated until the desired $u(f)$ maximum is obtained.

4.0 DIGITAL COMPUTER SOLUTIONS

4.1 General

Before problems of the type outlined in Section 3.4 can be solved by digital computer techniques, a concise mathematical statement must be formulated outlining all program requirements. The purpose of this statement is to give the programmer an insight into the type and scope of the problem to be solved by describing the algebraic relationships approximating the set of differential equations.

The format for this section includes the construction and usefulness of networks, required program inputs and outputs, the solution procedure, and a program flow chart. The type analyzed in Section 3.4.2 will be used to illustrate the manner of presenting the problem.

4.2 Mesh Networks

Once a configuration for the thermal protection system has been established, an array of regularly spaced straight lines that are mutually perpendicular to the heat-flow path are constructed within the boundaries starting from some arbitrary but fixed point $u(x_0, t_0)$, and positive number h , the mesh size for the variable x . In addition, a time mesh size p is chosen and the mesh consisting of another set of parallel lines constructed normal to the t coordinate. The method of selecting h and p is discussed in Section 4.4.

The network for the problem in Section 3.4.2 consists of three mesh as shown in Figure 15; one for each section of thermal insulation at $s \leq x \leq f$, $0 \leq x \leq r$, and $r \leq x \leq c$ for $t \geq 0$. Interior mesh points are those lying within the boundary mesh points at $t = 0$, and $x = s, f, 0, r$, and c .

Now the temperatures at all interior mesh points can be found by solving the appropriate difference equation mn times, where mn is merely the total number of interior mesh points. Boundary mesh point temperatures are either given or can be calculated from the relationships which apply at a particular interface.

4.3 Program Input

This particular portion of the problem statement includes all known data that are required to obtain a solution. Referring again to the case in Section 3.4.2, this information can be listed as given in paragraphs 4.3.1 through 4.3.11. If sufficient data are involved, presentation in tabular form might prove advantageous.

4.3.1 Surface Temperatures

The temperature of the surface is given as a function of time by listing a sufficient number of temperatures, $u(s)$, and related times, t , to adequately represent this function. Since an exact, concise mathematical formulation is usually not possible, interpolation can be employed to approximate values for $u(s, t)$, which lie between given points. Example:

Time increment, $\Delta t = 1$ minute

Total flight time, $t_T = 200$ minutes

Surface temperatures, $u(s) = 80.0^\circ\text{F}, 85.0^\circ\text{F}, 90^\circ\text{F} \dots$

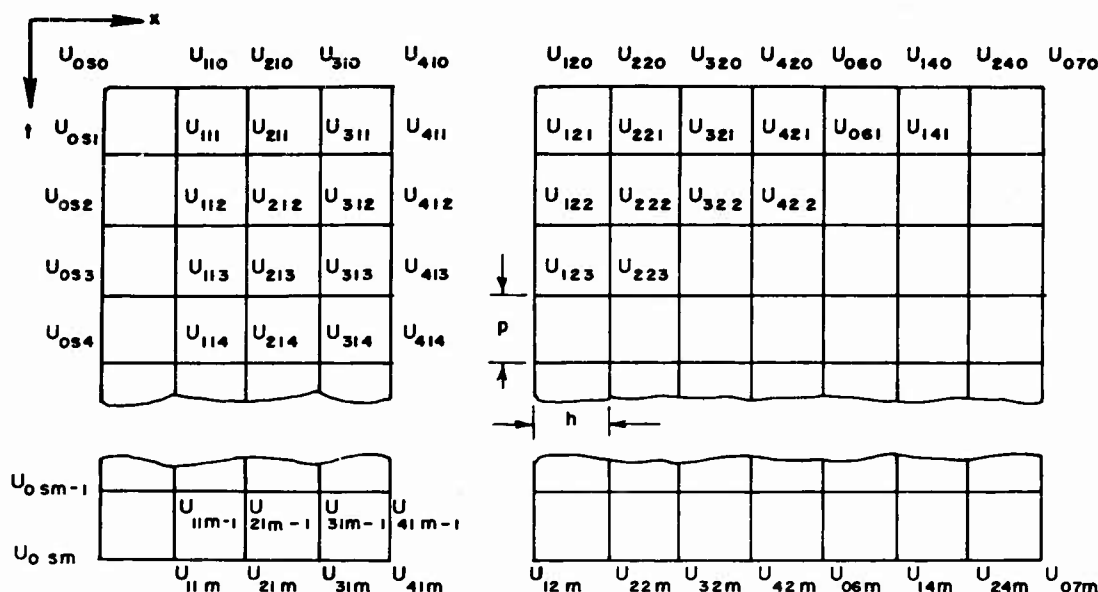


Figure 15. Mesh Network

4.3.2 Initial Temperatures

Since three mesh networks are necessary to depict the approximate solution for the case in Section 3.4.2, a general temperature term can be introduced denoting the space and time position associated with a given or calculated temperature. Therefore, $u(i,j,m)$ identifies a particular interior mesh point where i refers to the spatial mesh column in any mesh, j is a particular mesh, and m the time position or row in the mesh.

The initial temperatures can now be written as $u(i,j,0)$ and are given for all i,j .

4.3.3 Spatial Mesh Size

Sufficient accuracy will usually be obtained if the spatial mesh size, h , is made small. Therefore, the number of interior mesh points and the number of equations to be solved will generally be large. Some compromise is involved in selecting a value for h that will permit satisfactory accuracy with a reasonable number of interior mesh points.

4.3.4 Temperature at $x = r$

The temperatures of the spatial mesh column $x = r$ are equal to the saturation temperature of the coolant used as the expendable heat sink and is a function of ambient

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pressure, flight altitude, or the pressure within the system. Example:

For $0 \leq t \leq 150$:

Time increment, $\Delta t = 150$ minutes

Total time, $t_T = 150$ minutes

Temperature, $u(r, t) = 80^\circ\text{F}$

For $150 < t \leq 200$:

Time increment, $\Delta t = 5$ minutes

Total time, $t_T = 50$ minutes

Temperature, $u(r, t) = 80.0^\circ\text{F}, 85^\circ\text{F}, 90^\circ\text{F} \dots$

4.3.5 Densities

The density of each material used in the construction of the thermal protection system is as follows:

$$\rho_1, \rho_2, \rho_3, \dots$$

4.3.6 Thermal Conductivities

The thermal conductivity of each material used in the construction of the thermal protection system as a function of temperature is also required. Example:

$J = 1$:

$$u = 100^\circ\text{F}, 200^\circ\text{F}, 300^\circ\text{F} \dots$$

$$k(u) = 1.0, 1.1, 1.2 \dots$$

4.3.7 Specific Heat

The specific heat of each material used in the construction of the thermal protection system as a function of temperature is also required. Example:

$J = 1$:

$$u = 100^\circ\text{F}, 200^\circ\text{F}, 300^\circ\text{F} \dots$$

$$c_p(u) = 20.0, 20.1, 20.2 \dots$$

4.3.8 Temperature at $x = v$

The temperature of the compartment atmosphere is a known constant equal to $u(v)$.

4.3.9 Cross-sectional Area

Every member contributing to the internal flow of heat that does not have the same cross-sectional area as the representative element chosen is given as S_j .

4.3.10 Length of Heat Transfer Path

The thickness or length of every member in the direction of heat flow is given as L_j .

4.3.11 Coolant Enthalpy Change

The heat sink capability of the expendable coolant is a function of the ambient pressure at which the conversion from liquid to a gas takes place. This pressure can generally be assumed to be equal to that existing at flight altitudes; therefore, it can be related to flight time. Example:

For $0 \leq t \leq 150$ minutes:

Time increment, $\Delta t = 150$ minutes

Total time, $t_r = 150$ minutes

Enthalpy, $h_{fg} = 1050 \frac{\text{BTU}}{\text{lb}}$

For $150 < t \leq 200$:

Time increment, $\Delta t = 5$ minutes

Total time, $t_r = 50$ minutes

Enthalpy, $h_{fg} = 1049, 1047 \dots$

4.4 Program Output

This section lists the desired information resulting from an IBM 7090 Digital Computer solution of the problem that is identified in Section 3.4.2.

4.4.1 Time Mesh Size

The size of the time mesh required to satisfy stability criteria must be determined.

4.4.2 Thermal Property Coefficients

The coefficients of the expressions for thermal conductivity (A_{0j}, A_{1j}, A_{2j}) and specific heat (B_{0j}, B_{1j}) for each material are input requirements.

4.4.3 Temperature Printout

The temperature at spatial mesh points $s, f, o, r,$ and c at m times and the temperature at all spatial mesh points at the times indicated will be of interest; for example $(t_0 t_{15} t_{30} t_{100} t_{150} t_m)$.

4.4.4 Heat Removal Rates

The instantaneous rates of heat removed by the re-entry and compartment environmental cooling systems q_r and q_v at m times are needed to determine the Q_r and Q_v .

4.4.5 Total Heat Removed

The total amount of heat removed by the re-entry and compartment environmental cooling systems, Q_r and Q_v , are required to determine the coolant weights.

4.4.6 Total Amounts of Expended Coolant

The total weights of coolant expended by the re-entry and compartment environmental cooling systems, W_r and W_v , are used to calculate total system weights.

4.5 Problem Solution Procedure

This section describes a routine for determining the solution of the problem statement outlined in Section 3.4.2. If the operations are performed in the sequence listed, the temperature distributions, rates of heat transfer, total heat removed, and the weight of the thermal protection system can be calculated. Basically, the procedure is divided into two sections; first, a sub-routine, which generates preliminary information such as expressions for $k(u)$ and $c_p(u)$, the time mesh size, p , and the transformation of certain input data that are expressed as a function of flight time so that it coincides with the calculated time mesh size; and second, the order of solution of the finite difference equations derived in Sections 3.3 and 3.4. The step-by-step procedures are given in the subparagraphs that follow.

4.5.1 Thermal Properties

With the thermal conductivity k_j and the specific heat c_{pj} as a function of temperature given, determine the relationships

$$k_j = A_{0j} + A_{1j} u + A_{2j} u^2, \quad (123)$$

$$c_{pj} = B_{0j} + B_{1j} u \quad (124)$$

by the method of least squares as presented in Appendix II.

4.5.2 Time Mesh Size

4.5.2.1 From the input of surface temperatures as a function of flight time, pick out the maximum u_s .

4.5.2.2 Solve for the maximum k by inserting u_s maximum in place of u in the expression

$$k_i = A_{0i} + A_{1i} u + A_{2i} u^2 \quad (125)$$

4.5.2.3 Solve for the maximum c_p by inserting u_s maximum in place of u in the expression

$$c_{p_i} = B_{0i} + B_{1i} u \quad (126)$$

4.5.2.4 From the input of material densities, select ρ , and insert together with h , k_1 , and c_{p1} into

$$p_i = \frac{5 h \rho_i c_{p_i}}{2 k_i} \quad (127)$$

4.5.2.5 Repeat steps in paragraphs 4.5.2.2 through 4.5.2.4 for $j = 2$ to determine p_2 .

4.5.2.6 Select the smaller value, either p_1 or p_2 , and convert it to the nearest preceding integer, which is an exact divisor of the total flight time.

4.5.2.7 Transform the initial set of surface temperatures, $u(s, t)$ into a new set with a time increment equal to p by the process of interpolation.

4.5.2.8 Repeat step in Section 4.5.2.7 for $u(r, t)$ and $h_{fg}(r, t)$.

4.5.3 Solution of Finite Difference Equations

4.5.3.1 Determine the temperatures for all interior mesh points U_{11}^* , U_{21} , ..., U_{n1} ; U_{12} , U_{22} , ..., U_{n2} ; U_{13} , U_{23} , ..., U_{n3} at time $m = 1$ from the following equation:

$$U_{ijm+1} = U_{ijm} + \frac{p}{20 \rho_j h^2 (B_{0j} + B_{1j} U_{ijm})} \left[(A_{1j} + 2 A_{2j} U_{ijm}) \right. \\ \left. (U_{i+1,j,m}^2 - 2 U_{i+1,j,m} U_{i-1,j,m} + U_{i-1,j,m}^2) + 4 (A_{0j} + A_{1j} U_{ijm} + \right. \\ \left. A_{2j} U_{ijm}^2) (U_{i+1,j,m} - 2 U_{ijm} + U_{i-1,j,m}) \right] \quad (128)$$

* Temperatures at specific locations are denoted by U in this section.

4.5.3.2 Determine the temperature at the exterior mesh points $x = f$ and $x = o$ at time $m = 1$ from the following equations:

$$\left(C_1 + C_3 + \frac{C_2}{h}\right) U_{f,m+1} - C_3 U_{o,m+1} = \frac{C_2}{h} U_{f-1,m+1} + C_1 U_{s,m+1} - C_4, \quad (129)$$

$$C_3 U_{f,m+1} - \left(C_3 + \frac{C_5}{h}\right) U_{o,m+1} = -C_4 - \frac{C_5}{h} U_{o+1,m+1}, \quad (130)$$

where

$$C_1 = \frac{S_4}{4 \ell_4} \left(4A_{04} + 2A_{14} U_{s,m+1} + 2A_{14} U_{f,m} + A_{24} U_{s,m+1}^2 + 2A_{24} U_{s,m+1} U_{f,m} + A_{24} U_{f,m}^2 \right), \quad (131)$$

$$C_2 = A_0 + \frac{A_1}{2} U_{f-1,m+1} + \frac{A_1}{2} U_{f,m} + \frac{A_2}{4} U_{f-1,m} + \frac{A_2}{2} U_{f-1,m+1} U_{f,m} + \frac{A_2}{4} U_{f,m}^2, \quad (132)$$

$$C_3 = \frac{1}{4 \ell_5} \left(4A_{05} + 2A_{15} U_{o,m} + 2A_{15} U_{o,m+1} + A_{25} U_{f,m}^2 + 2A_{25} U_{f,m} U_{o,m} + A_{25} U_{o,m}^2 \right), \quad (133)$$

$$C_4 = \sigma F_\epsilon F_a \left(U_{f,m}^4 + 1838.8 U_{f,m}^3 + 1267.9 \times 10^3 U_{f,m}^2 + 3885.6 \times 10^5 U_{f,m} - U_{o,m}^4 - 1838.8 U_{o,m}^3 - 1267.9 \times 10^3 U_{o,m}^2 - 3885.6 \times 10^5 U_{o,m} \right), \quad (134)$$

$$C_5 = A_0 + \frac{A_1}{2} U_{o,m} + \frac{A_1}{2} U_{o+1,m+1} + \frac{A_2}{4} U_{o,m}^2 + \frac{A_2}{2} U_{o,m} U_{o+1,m+1} + \frac{A_2}{4} U_{o+1,m+1}^2 \quad (135)$$

4.5.3.3 Determine the rate of heat transfer at time $m = 1$ to and from and the rate of heat storage at the exterior mesh point $x = r$. Now the rate of heat transfer to r

$$q_{is} = \frac{1}{4h} \left(4A_0 U_{r-1,m+1} - 4A_0 U_{r,m+1} + 2A_1 U_{r-1,m+1}^2 - 2A_1 U_{r,m+1}^2 + A_2 U_{r-1,m+1}^3 + A_2 U_{r-1,m+1}^2 U_{r,m+1} - A_2 U_{r-1,m+1} U_{r,m+1}^2 - A_2 U_{r,m+1}^3 \right), \quad (136)$$

$$q_{os} = \frac{1}{4h} \left(4A_0 U_{r,m+1} - 4A_0 U_{r+1,m+1} + 2A_1 U_{r,m+1}^2 - 2A_1 U_{r+1,m+1}^2 + A_2 U_{r,m+1}^3 + A_2 U_{r,m+1}^2 U_{r+1,m+1} - A_2 U_{r,m+1} U_{r+1,m+1}^2 - A_2 U_{r+1,m+1}^3 \right), \quad (137)$$

$$q_{ss} = \frac{5 \rho_s c p_s \ell_s S_s}{p} (U_{r, m+1} - U_{r, m}) \quad (138)$$

where $U_{r, m+1}$ is given.

4.5.3.4 Evaluate conditions at $x=r$ and time $m+1$. Three possibilities exist. If

$$a. \quad q_{is} = q_{os} + q_{ss} \quad (139)$$

then no heat is removed by the cooling system, or $q_{d5} = 0$, and $U_{r, m+1} = U_{r, m+1}$ (given by input).

$$b. \text{ When } q_{is} > q_{os} + q_{ss} \quad (140)$$

then the heat removed by the cooling system is

$$q_{d5} = q_{is} - (q_{os} + q_{ss}) \quad (141)$$

and $U_{r, m+1} = U_{r, m+1}$ (given by input).

$$c. \text{ Finally, if } q_{is} < q_{os} + q_{ss} \quad (142)$$

then no heat is removed by the cooling system ($q_{d5} = 0$) and $U_{r, m+1} \neq U_{r, m+1}$ (given by input). When condition (c) applies a new $U_{r, m+1}$ must be determined from the following relationship, converting Equation (142) to an equality

$$\begin{aligned} & \frac{A_2}{2h} U_{r, m+1}^3 + \left[\frac{A_2}{2h} (U_{r-1, m+1} + U_{r+1, m+1}) \right] U_{r, m+1}^2 + \left(\frac{5 \rho_s c p_s \ell_s S_s}{p} + \right. \\ & \left. \frac{2A_0}{h} + \frac{A_1}{h} - \frac{A_2}{4h} U_{r-1, m+1}^2 - \frac{A_2}{4h} U_{r+1, m+1}^2 \right) U_{r, m+1} - \left(\frac{A_0}{h} U_{r-1, m+1} + \right. \\ & \left. \frac{A_1}{2h} U_{r-1, m+1}^2 + \frac{A_2}{4h} U_{r-1, m+1}^3 + \frac{A_0}{h} U_{r+1, m+1} + \frac{A_2}{4h} U_{r+1, m+1}^3 + \right. \\ & \left. \frac{5 \rho_s c p_s \ell_s S_s}{p} U_{r, m} \right) = 0 \quad (143) \end{aligned}$$

4.5.3.5 Determine the temperature at the external mesh point $x=c$ for $m+1$ from the following equation:

$$U_{c, m+1} = \frac{1}{C_1 + C_2 + C_h} (C_1 U_{c-1, m+1} + C_2 U_{c, m} + C_h U_v) \quad (144)$$

where

$$\begin{aligned} C_1 = \frac{1}{h} & \left(A_0 + \frac{A_1}{2} U_{c-1, m+1} + \frac{A_1}{2} U_{c, m} + \frac{A_2}{4} U_{c-1, m+1}^2 + \right. \\ & \left. \frac{A_2}{2} U_{c-1, m+1} U_{c, m} + \frac{A_2}{4} U_{c, m}^2 \right) \quad (145) \end{aligned}$$

$$C_2 = \frac{5 \rho_7 c_{p7} \ell_7}{p} \quad (146)$$

and C_h is the film heat-transfer coefficient at the wall.

4.5.3.6 Determine rates of heat transfer to the re-entry cooling system, q_r , and compartment environmental control system, q_v , at time $m = 1$. The former was calculated in Section 4.5.3.3. Now q_v can be established from the following:

$$q_{v,m+1} = C_h (U_{c,m+1} - U_v) \quad (147)$$

4.5.3.7 Determine temperatures at all mesh points and heat-transfer rates q_{d5} and q_{v7} for times $m = 2, 3, 4, \dots, n$ by repeating steps in Sections 4.5.3.1 through 4.5.3.6.

4.5.3.8 Determine the total heat removed at mesh points $x = r$ and $x = v$ during the mission thus

$$Q_r = np \sum_{m=1}^n q_r \quad (148)$$

and

$$Q_v = np \sum_{m=1}^n q_v \quad (149)$$

4.5.3.9 Determine the weight of the coolant expended during the mission at mesh points $x = r$ and $x = v$ thus

$$W_r = np \sum_{m=1}^n \frac{q_r}{h_{fg_r}} \quad (150)$$

$$W_v = np \sum_{m=1}^n \frac{q_v}{h_{fg_r}} \quad (151)$$

This completes the solution of the problem.

4.6 Program Flow Chart

The program flow chart presents in a concise manner a step-by-step solution procedure for a complex problem involving a number of operations. Two flow charts outlining the manner of solving the relationships for the case in Section 3.4.2 are shown in Figures 16 and 17. When the path connecting the events is traced, the sequence of introducing or storing terms is indicated as well as the origin of each input. A clearer picture of the method of solution may be obtained if the flow chart of the program is used in conjunction with the mesh network.

At this point some comments regarding the solution of certain of the equations in Section 4.5 should be in order. In general, the majority of the equations involve the straightforward application of the principles of algebra that result in a rather obvious solution. On the other hand, the solution of a polynomial, such as Equation (143), involves some knowledge of the nature of the one valuable root. Of course, the root of interest must be real; however, this condition by itself is certainly not a sufficient one since it is entirely feasible that the solution would yield more than one real root. Therefore, a second condition should be included to further restrict the number of possible results. A possibility here is to, in some fashion, bound the solution. For instance, if some insight into a limited range for the one valuable root can be applied by confining the neighborhood of the solution within finite limits, the probability that more than one real root will lie within these bounds is low. In fact, the extremes usually can be defined so that one valuable root will almost always be the only one that satisfies both of the previously mentioned restrictions.

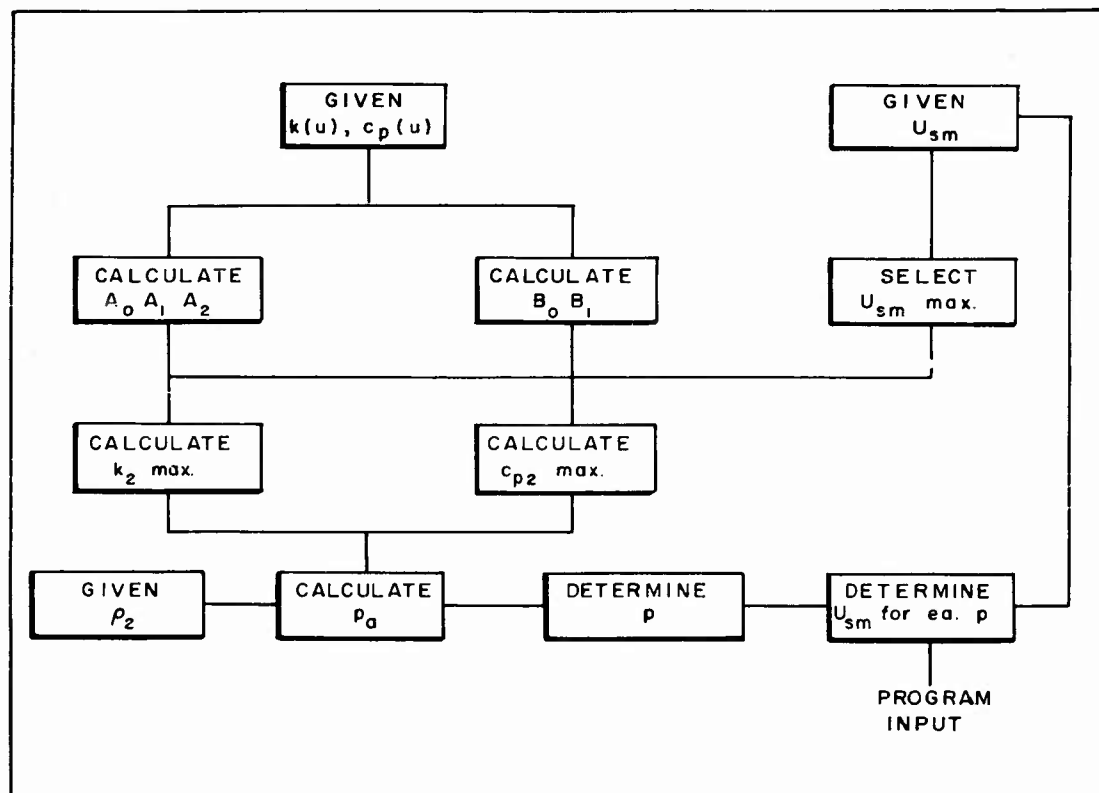


Figure 16. Program Flow Chart for Subroutine

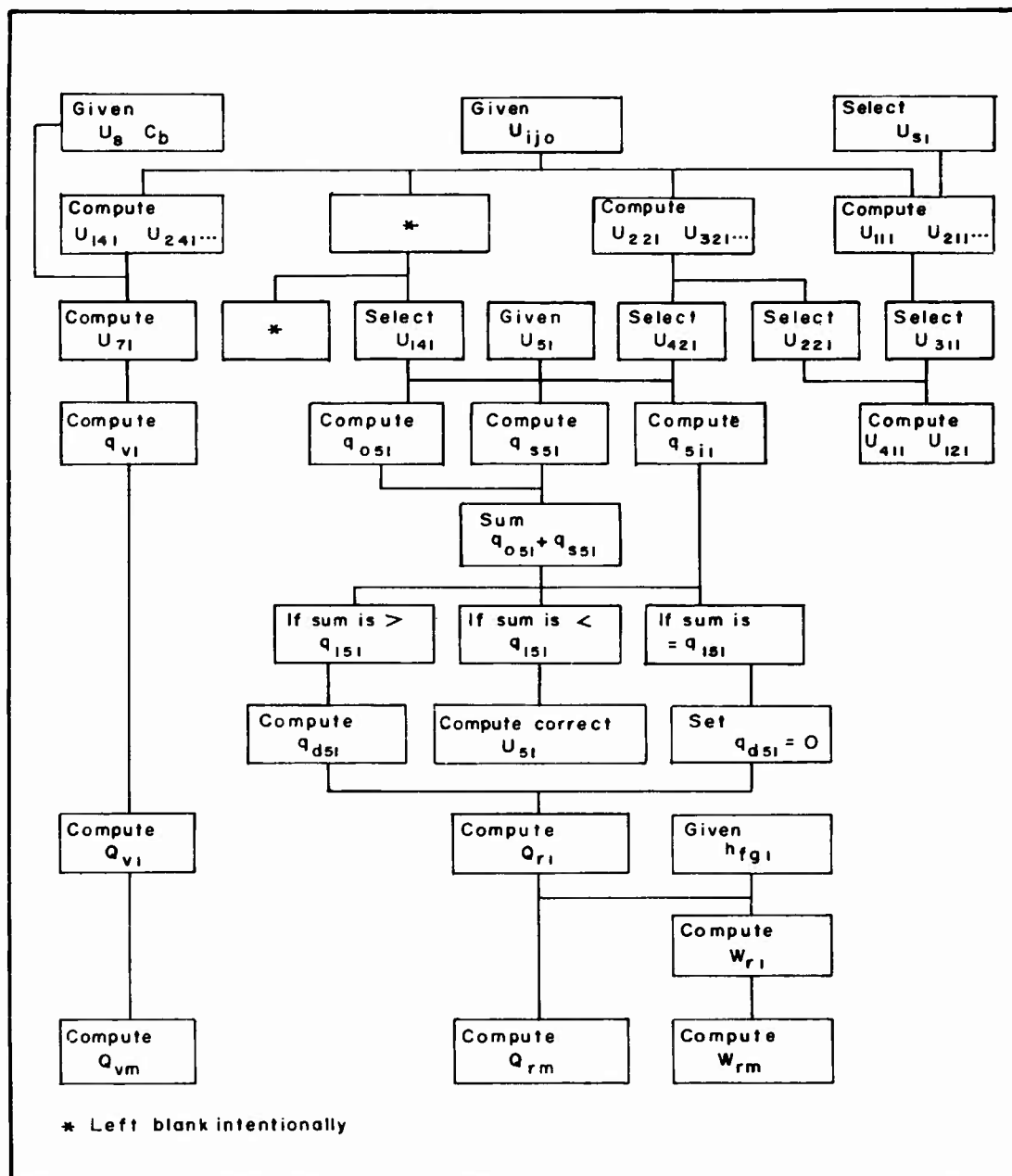


Figure 17. Program Flow Chart for the Insulated and Cooled Compartment

5.0 RESULTS

5.1 System Thermal Analysis

5.1.1 General

Numerical solutions for each of the three problems presented in Section 3 would represent a considerable effort in view of the number of parameters involved. Such a treatment of each case is not within the scope of this section; instead, the application of the finite difference equations and the method of optimizing the design of the thermal protection system will be illustrated for only one case.

The example selected was that designated in Section 3.4.1 "The Insulated and Cooled Structure." This problem is considered a compromise between the lengthy and, possibly, less meaningful solutions that are required for the case in Section 3.4.2, and the more simplified, straightforward approach applying to the case in Section 3.4.3. To clarify the results, we modified the case in Section 3.4.1 by terminating the solution at $x = c$ (see Figure 11). In addition, a thermal-conductance term that depended on temperature was introduced to include the effects of an insulation package on the heat transmitted to the cooling system. As a result, Equation (28) was modified to incorporate an additional term.

5.1.2 Program Inputs

This section lists most of the program inputs in tabular form as suggested in Section 4.3. In the case of surface temperatures of the vehicle during re-entry, these temperatures can be presented more conveniently in a graphical manner because of the number involved. Optimum insulation thicknesses will be determined for four histories of surface temperatures $U(s,t)$, shown in Figure 18, which are typical of those experienced at various locations on the outer heat shield. The number of curves and the maximum temperature of each were selected to enable a parametric survey and to facilitate the presentation of data. A listing of the IBM program input data is available in Table 1.

This table includes the thermal property data determined from the physical makeup of the thermal protection system. Coefficients for all polynomials were then determined by the method in Appendix II. The origin of most of this information is included in the references.

Three thicknesses of insulation were selected as a minimum for demonstrating the influence of this parameter on the weight of the system. The smallest thickness, arbitrarily chosen as 0.2 inch, was established in view of the magnitude of the spatial mesh size. The time mesh size was calculated by employing the stability criteria discussed in Section 4.5.2. Once this was established, the surface-temperature histories, U_{s1} through U_{s4} , could be tabulated by listing the temperatures that corresponded to each time step.

This completed the input requirements for solving the problem.

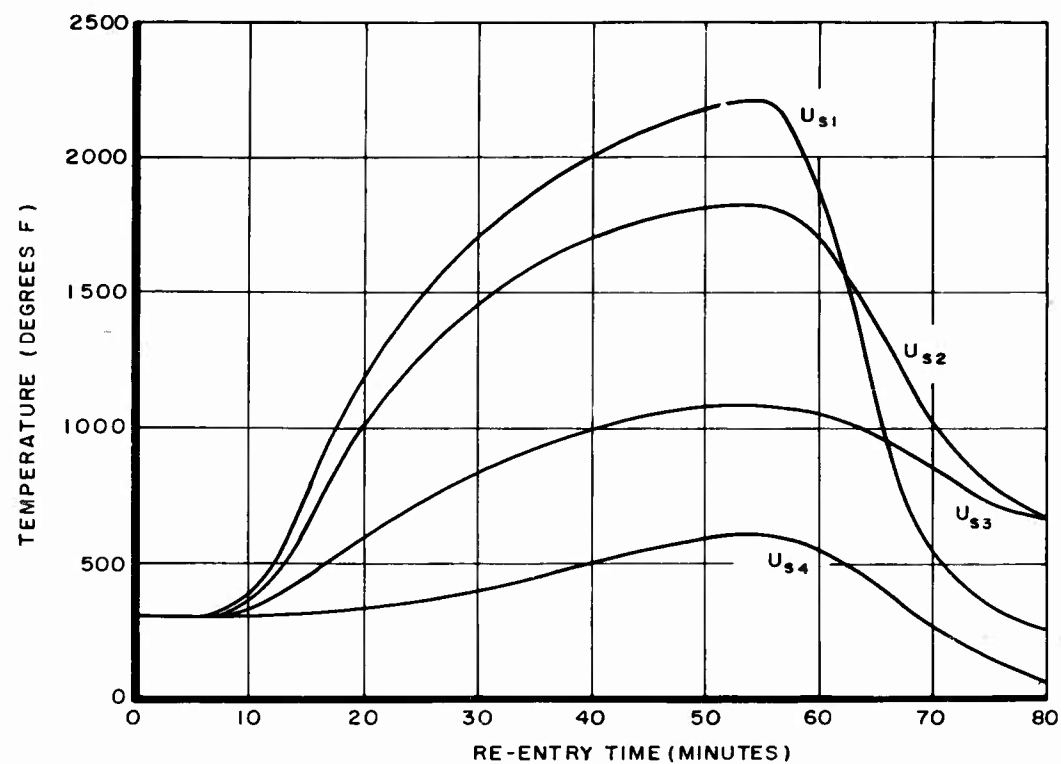


Figure 18. History of Surface-Equilibrium Temperature of Vehicle

TABLE 1
PROGRAM INPUTS

Inputs		Insulation Thickness (inches)		
		0.6	0.8	1.0
Number of Internal Mesh Points (i)		6	7	8
Coefficients of the Polynomial $k(u)$ (Insulation):	A_0	0.14	0.14	0.14
	A_1	-0.177×10^{-3}	-0.177×10^{-3}	-0.177×10^{-3}
	A_2	0.103×10^{-6}	0.103×10^{-6}	0.103×10^{-6}
Coefficients of the Polynomial $c_p(u)$	B_0	0.23	0.23	0.23
	B_1	0	0	0
Insulation Density (ρ)		12	12	12
Air Space Conductance $C_a(u)$	$\frac{A_{0a}}{1}$	0.45	0.45	0.45
	$\frac{A_{1a}}{1}$	0.4133×10^{-3}	0.4133×10^{-3}	0.4133×10^{-3}
Heat Shield Conductance $C_b(u)$	D_0	15.642	15.642	15.642
	D_1	-19.5×10^{-3}	-19.5×10^{-3}	-19.5×10^{-3}
	D_2	12.61×10^{-6}	12.61×10^{-6}	12.61×10^{-6}
Insulation Package Conductance $C_f(u)$	E_0	5.33×10^{-2}	4.0×10^{-2}	3.2×10^{-2}
	E_1	8.667×10^{-5}	6.5×10^{-5}	5.2×10^{-5}
	E_2	0	0	0
$\sigma F \epsilon$		2.884×10^{-10}	2.884×10^{-10}	2.884×10^{-10}
Spatial Mesh Size (h)		0.2	0.2	0.2
Time Mesh Size (p)		0.5	0.5	0.5
Number of Time Steps (m)		161	161	161
Initial Temperatures (except at $x = c$) U_{x0}		70	70	70
Temperature at $x = c$ (U_{c0}, U_{c1}, \dots)		200	200	200

5.1.3 Program Output

Numerical solutions to the problem were obtained by following a procedure similar to the one outlined in Section 4.5, which led to many of the outputs suggested in Section 4.4.

The first results are graphical representations of the temperature distribution through two thicknesses of insulation, 0.6 and 1.0 inch, at the re-entry times specified. The surface temperature history, which applies in each case, has been noted and the curves are presented in Figures 19 through 26.

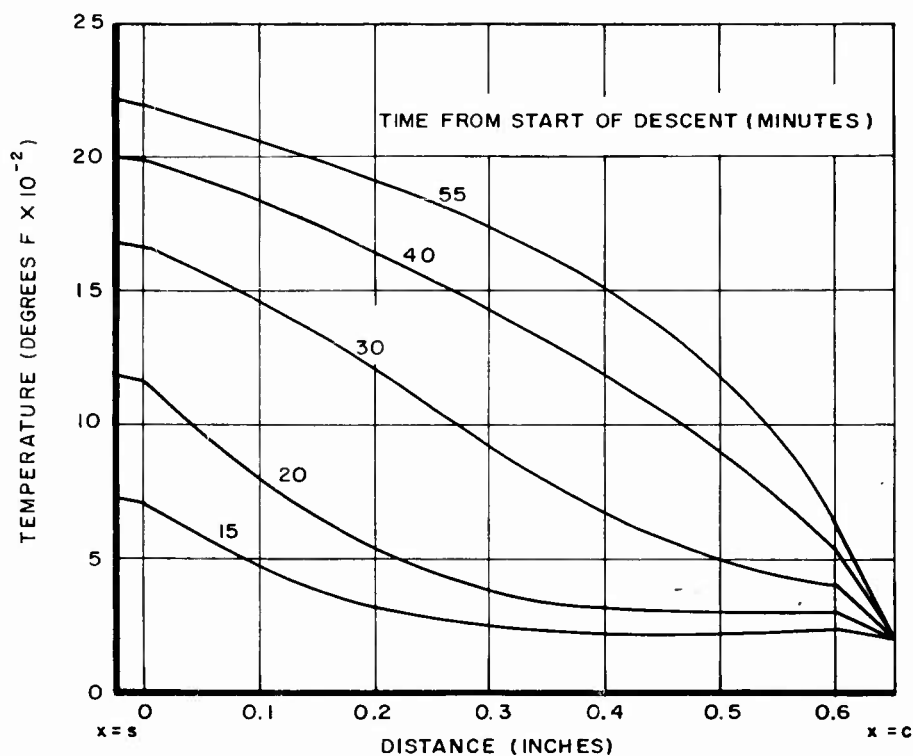


Figure 19. Temperature Gradient Through Thermal Protection System for Insulation Thickness of 0.6 Inch (Surface Temperature U_{s1})

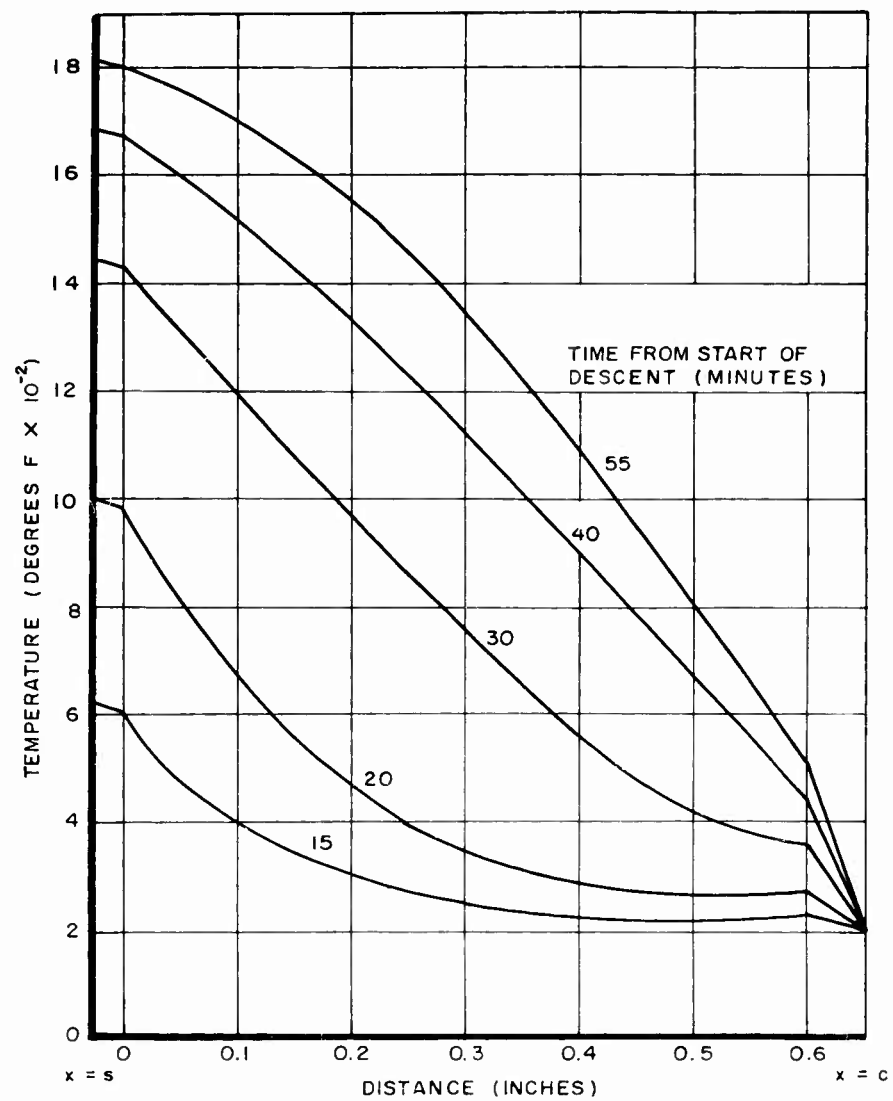


Figure 20. Temperature Gradient Through Thermal Protection System for Insulation Thickness of 0.6 Inch (Surface Temperature U_{s2})

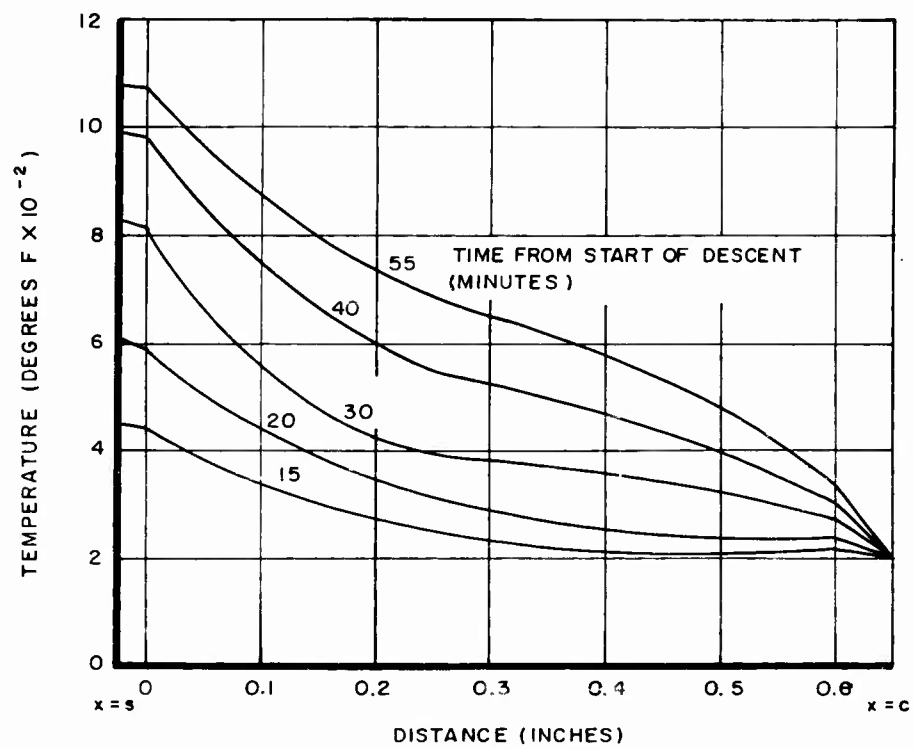


Figure 21. Temperature Gradient Through Thermal Protection System for Insulation Thickness of 0.6 Inch (Surface Temperature U_{s3})

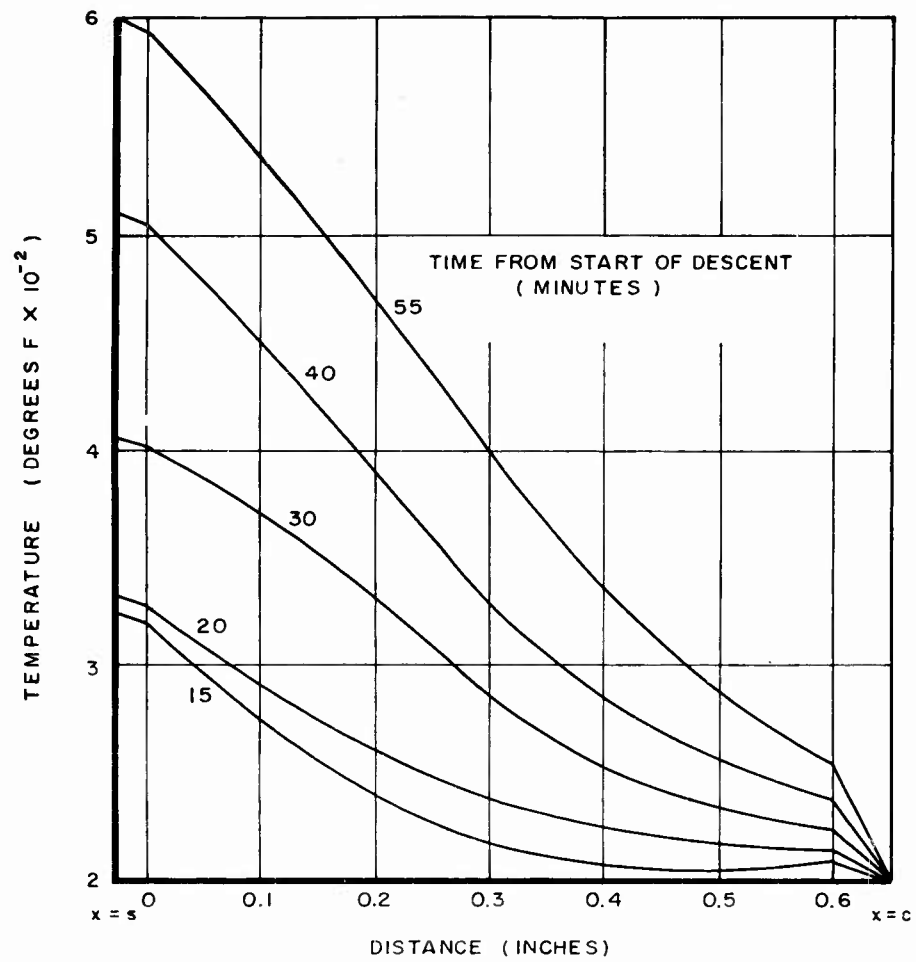


Figure 22. Temperature Gradient Through Thermal Protection System for Insulation Thickness of 0.6 Inch (Surface Temperature U_{s4})

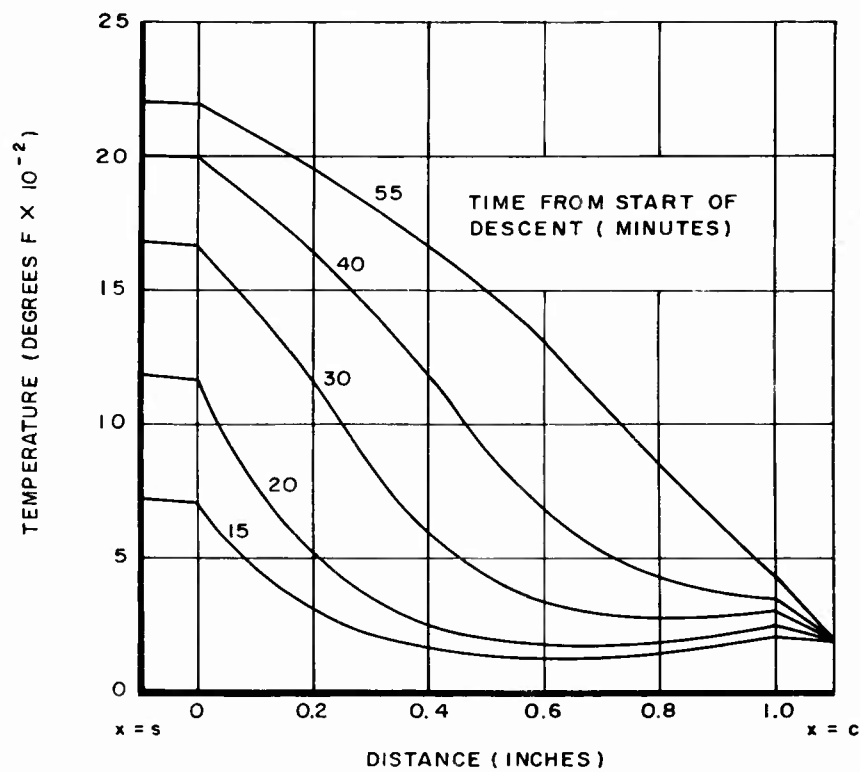


Figure 23. Temperature Gradient Through Thermal Protection System for Insulation Thickness of 1.0 Inch (Surface Temperature U_{s1})

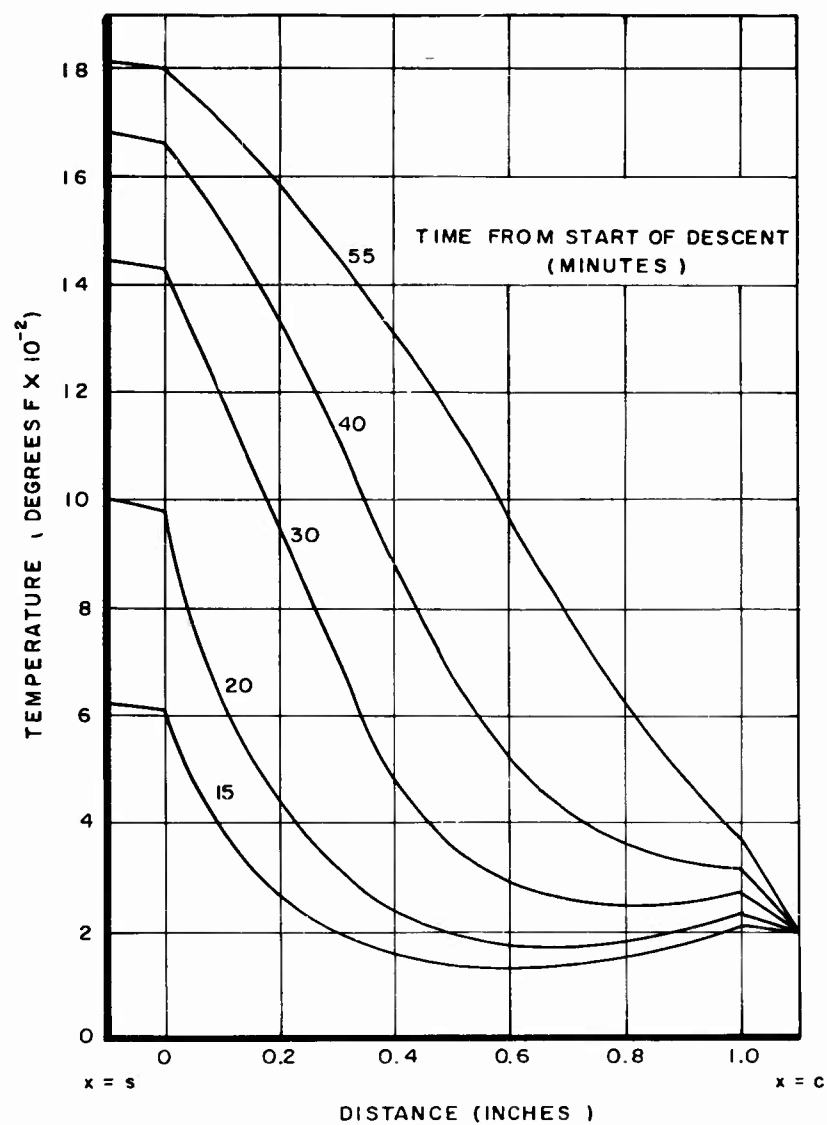


Figure 24. Temperature Gradient Through Thermal Protection System for Insulation Thickness of 1.0 Inch (Surface Temperature U_{s2})

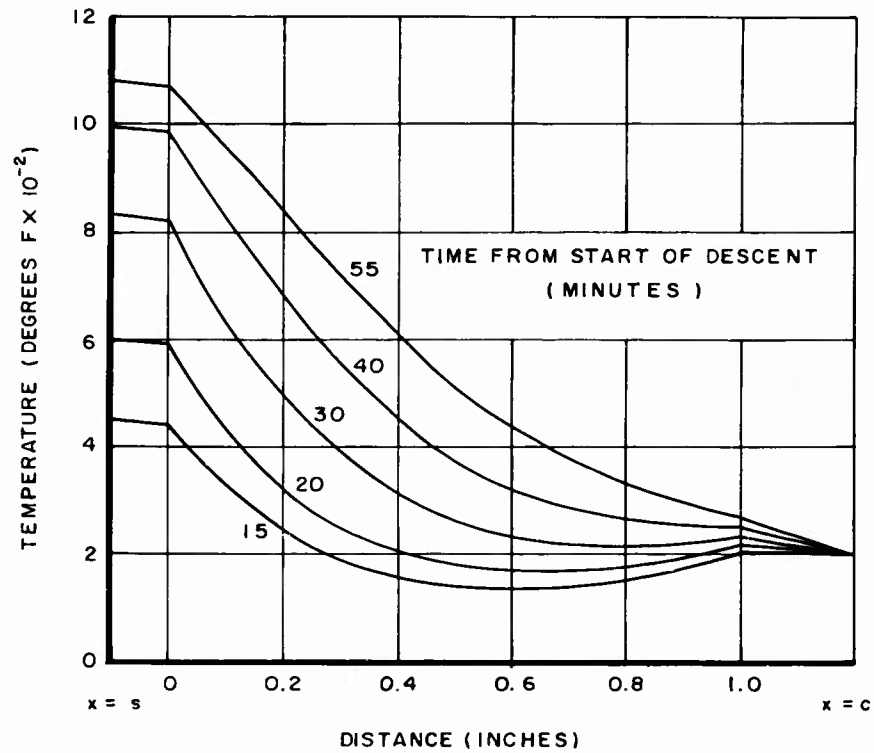


Figure 25. Temperature Gradient Through Thermal Protection System for Insulation Thickness of 1.0 Inch (Surface Temperature U_{s3})

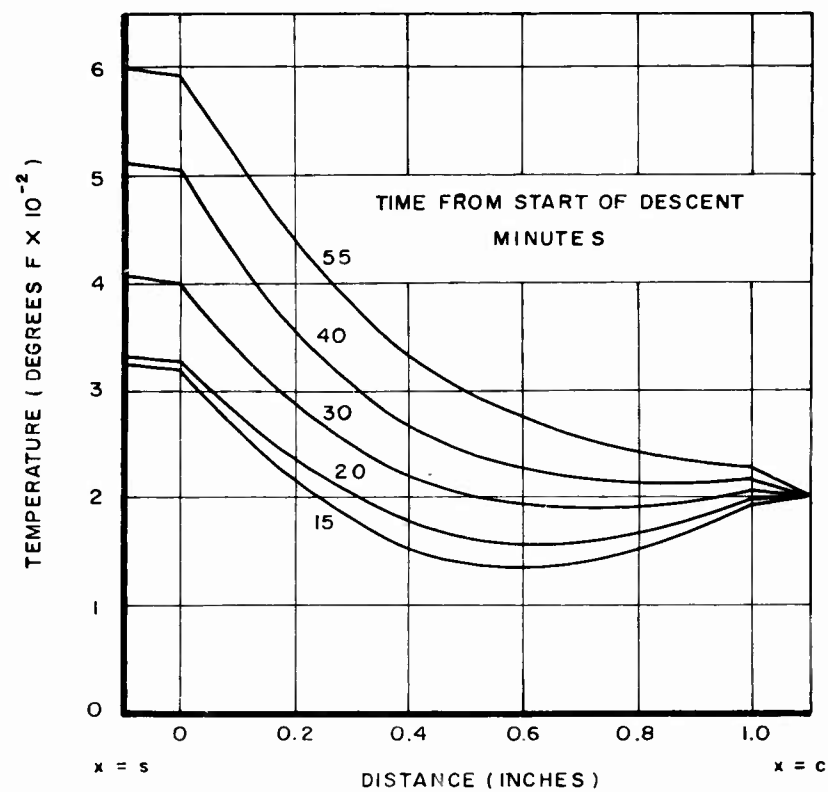


Figure 26. Temperature Gradient Through Thermal Protection System for Insulation Thickness of 1.0 Inch (Surface Temperature U_{s4})

Several interesting observations can be made from these results. For 15 minutes after the start of re-entry, we note that the temperature gradient is zero at some point within the insulation. On either side of this point, the gradient indicates that heat is conducted from both the outside and the inside faces of the insulation to internal elements. Once the temperature of the external surface begins to increase, heat is conducted through the heat shield and the edges of the insulation package to the inside face of the insulation. The response of this path is very rapid due to the negligible heat capacity of each of these components. On the other hand, the heat capacity of the insulation cannot be neglected; therefore, the transfer gradient through the insulation lags the transfer gradient through the heat shield supports. Simultaneous conduction along both paths gives the appearance of a heat generation term at the inside face of the insulation.

As the time from the start of re-entry increases, the point of zero temperature gradient moves to the right until it passes through the insulation. At this time, heat is being transferred in one direction only.

The assumption that the temperature of the primary structure is initially 130°F above the temperature of the insulation influences two directional heat flow. This assumption does not affect the temperature distribution beyond a 15-minute period for all but one case, and, in general, any contribution of this assumption to the results is rather insignificant.

One will also note that the heat shield plays a very small role in attenuating heat transfer to the primary structure. As mentioned previously, its major function is to transfer aerodynamically generated heat back across the boundary layer. The equivalent conductance of this honeycomb panel is relatively high; hence, the temperature gradient across its thickness is small.

In comparison, the air space between the insulation and the primary structure offers considerable resistance to the flow of heat as indicated by the temperature drop between these two surfaces.

The rates of heat transfer to the cooling system are shown in Figures 27 through 30 for each thickness of insulation as a function of re-entry time. For each, the coolant requirements are computed from the value of the integral under each curve. The amounts of heat absorbed for each computer run are given in Table 2.

TABLE 2
HEAT ABSORBED FOR COMPUTER RUNS

Insulation Thickness (inches)	Total Heat Absorbed (BTU per Sq Ft)			
	Temperature History			
	U_{s1}	U_{s2}	U_{s3}	U_{s4}
0.6	454	355	149	44
0.8	325	248	101	27
1.0	232	174	68	15

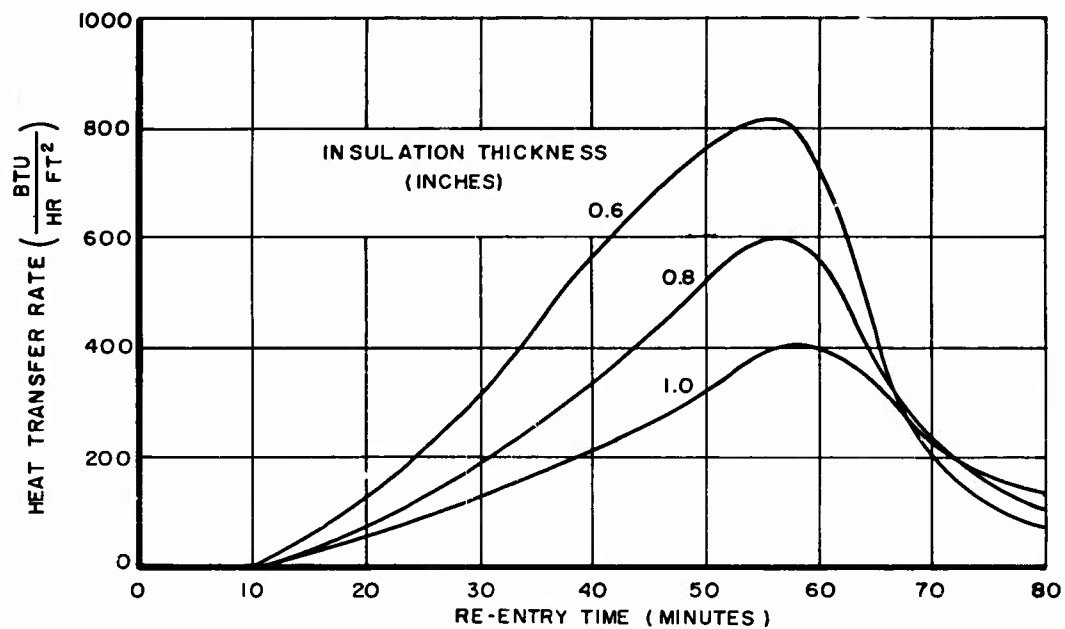


Figure 27. Effect of Insulation Thickness on Rate of Heat Transfer to Coolant (Surface Temperature U_{s1})

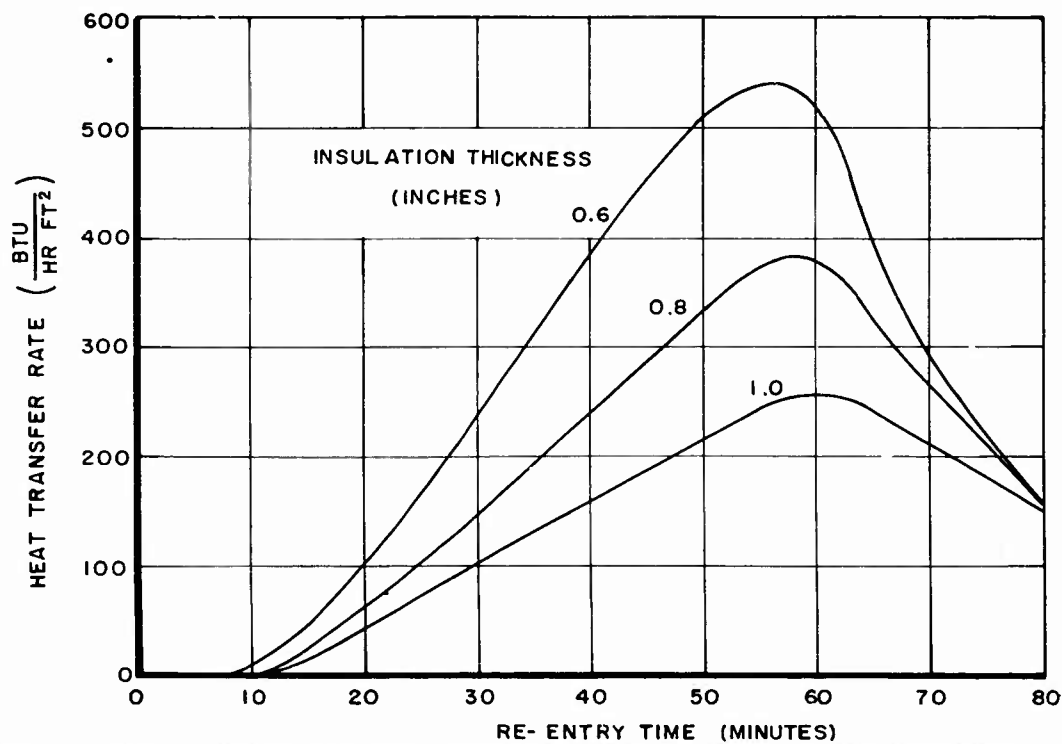


Figure 28. Effect of Insulation Thickness on Rate of Heat Transfer to Coolant (Surface Temperature U_{s2})

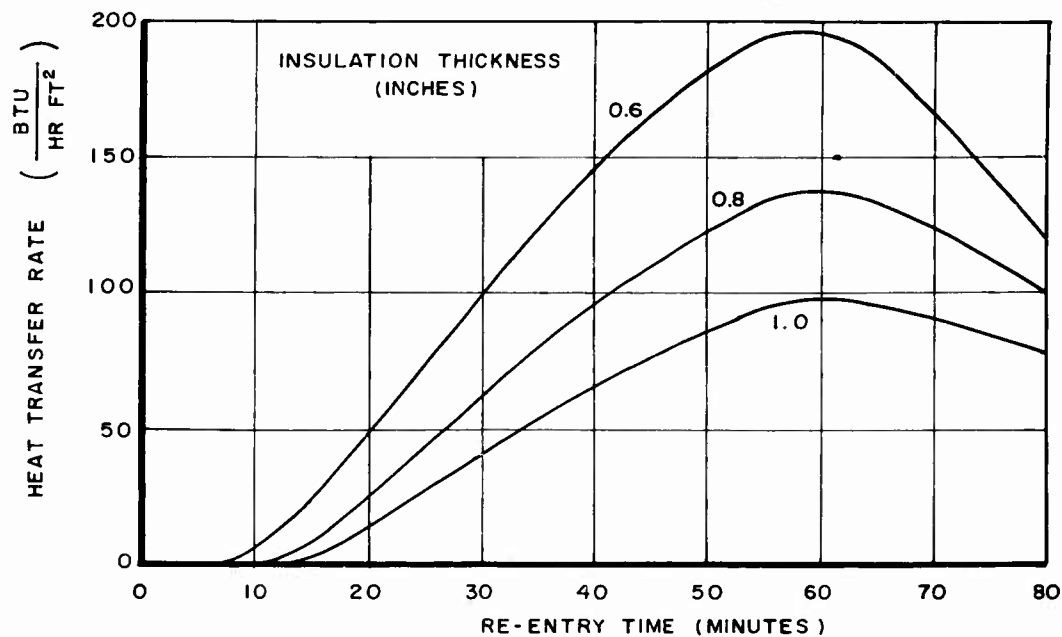


Figure 29. Effect of Insulation Thickness on Rate of Heat Transfer to Coolant Surface Temperature U_{s3}

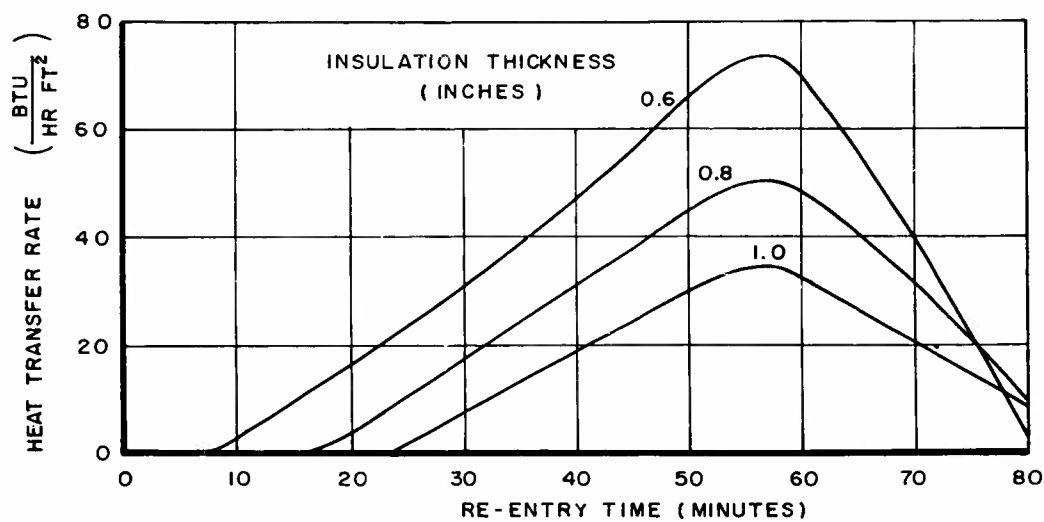


Figure 30. Effect of Insulation Thickness on Rate of Heat Transfer to Coolant Surface Temperature U_{s4}

The maximum rate of heat transfer is shown in Figure 31 and the coolant requirements are shown in Figure 32 as each is affected by the insulation thickness and the maximum equilibrium temperature. The use of this latter term as a parameter was justified, since all temperature histories are of the same duration and the general shape of the curves is similar.

Figure 32 indicates the heat transmitted to the coolant is nearly zero if the maximum surface equilibrium temperature is not high for some insulation thicknesses. For this case, the elimination of the cooling system might be considered.

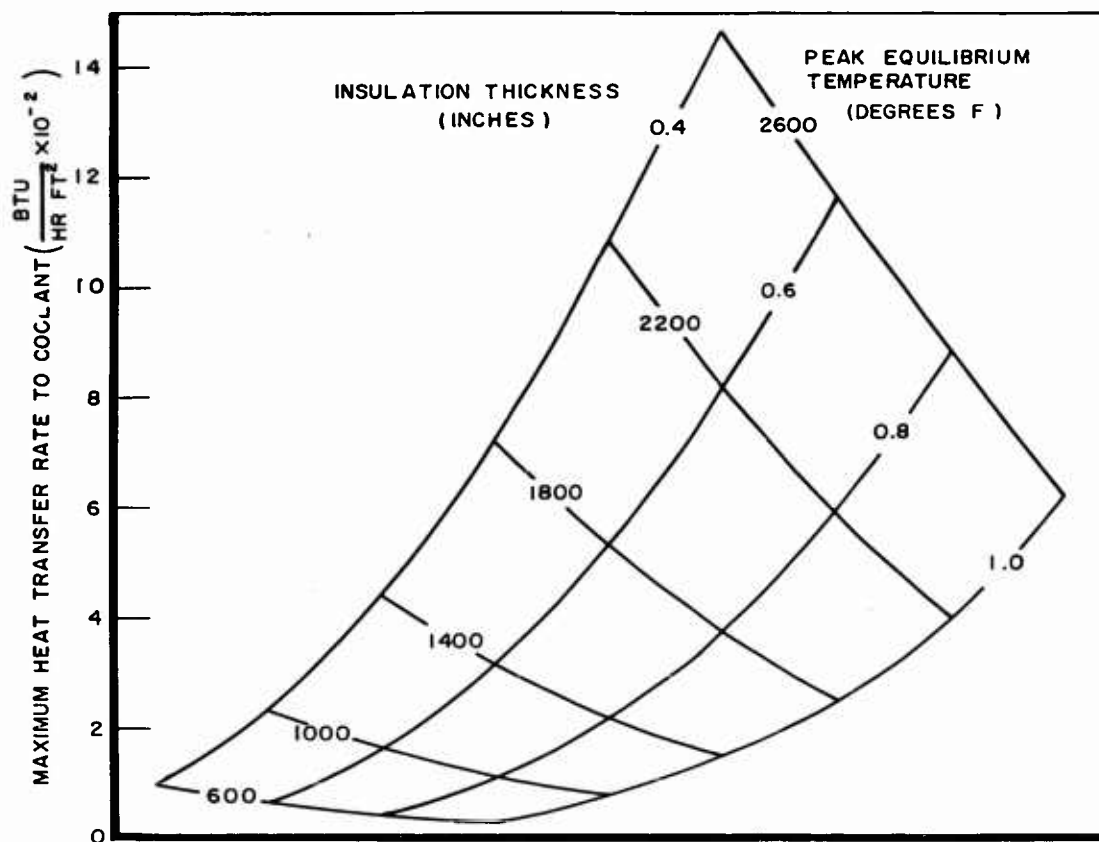


Figure 31. Effect of Temperature and Insulation Thickness on Rate of Heat Transfer

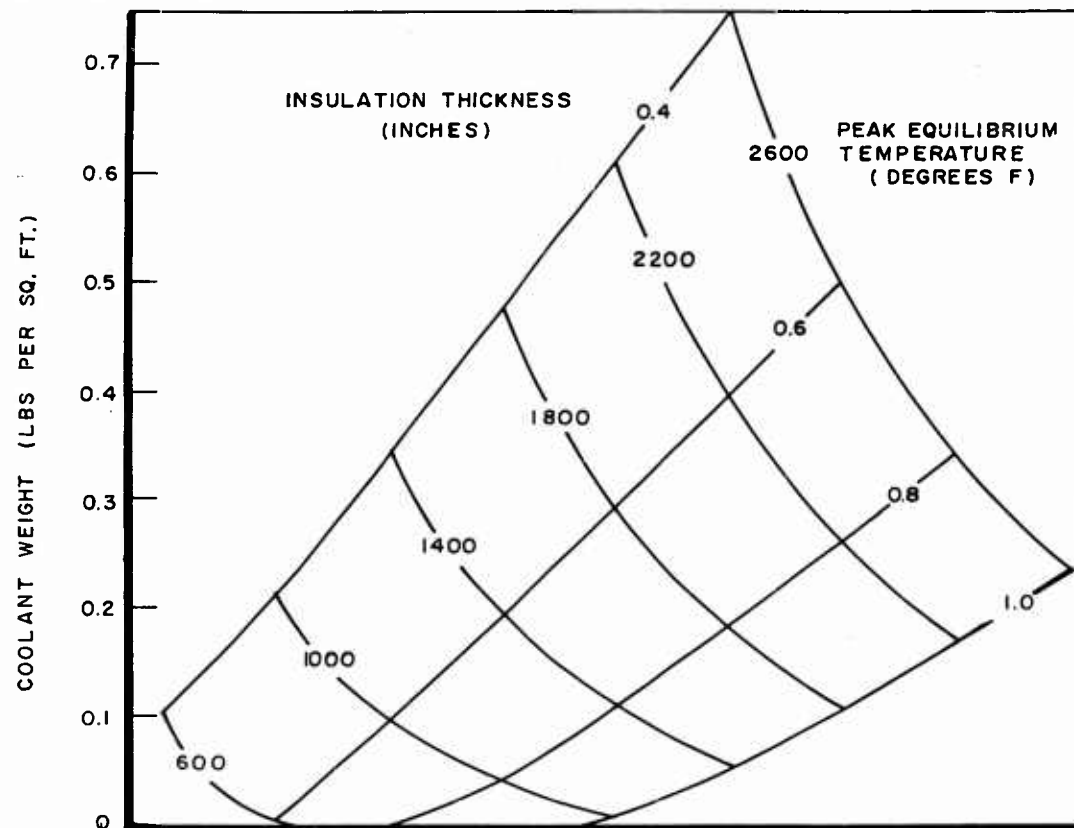


Figure 32. Effect of Temperature and Insulation Thickness on Coolant Weight

5.2 Insulation and Coolant Tradeoff

If the coolant requirements, as shown in Figure 32, are combined with the weight of the insulation, a relationship between the total weight of the thermal protection system and the thickness of insulation can be demonstrated for a number of outer surface temperatures. These results are shown in Figure 33 in which the maximum surface temperature is used as a notation of the particular history (refer to Figure 18). An examination of these results reveals that the selection of a thickness of insulation can be made, which, when combined with the amount of coolant required, will result in a thermal protection system of minimum weight.

This type of presentation may be conveniently used as design information, since the surface of the re-entry vehicle can be sectioned into temperature zones. The boundaries of these zones can be judiciously assigned, based on temperature distributions and the surface temperature history for the re-entry trajectory, which results in maximum heating. In the case of glide vehicles, temperature gradients on the surface are relatively insignificant; hence, the total number of zones would be few and can be based on a modest deviation from an average temperature. As a result, a procedure is established that finds

application as a practical design tool. The more problematical aspects are those that are associated with the derivation of analytical relationships and the programming of these in each case. These areas were emphasized in this study.

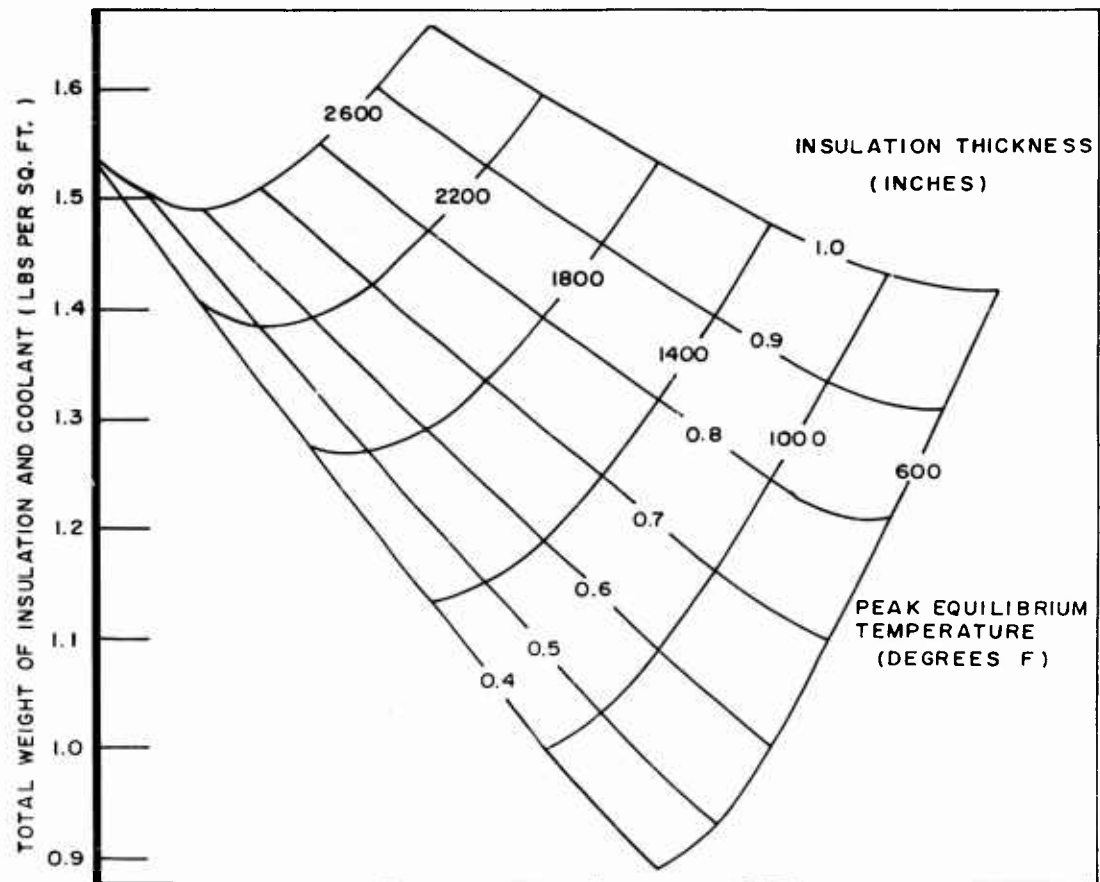


Figure 33. Insulation - Coolant Weight Tradeoff

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APPENDIX I

DERIVATION OF FINITE DIFFERENCE RELATIONSHIPS

In solving a differential equation of the form in Equation (13), one can express conveniently the temperature distribution, U_x , as an n^{th} degree polynomial containing $(n+1)$ arbitrary constants as

$$U_x = a_0 + a_1 x^{(1)} + a_2 x^{(2)} + a_3 x^{(3)} + \dots + a_n x^{(n)} \quad (152)$$

If x is replaced by the variable

$$v = \frac{x - x_0}{h} \quad (153)$$

where h is the interval between two successive values of the argument x , the differences of U_v may now be used to determine the values of the coefficients $a_0, a_1, a_2, \dots, a_n$ thus

$$\left. \begin{aligned} \Delta U_v &= a_1 + 2a_2 v^{(1)} + 3a_3 v^{(2)} + \dots + na_n v^{(n-1)} \\ \Delta^2 U_v &= 2 \cdot 1 \cdot a_2 + 3 \cdot 2 \cdot a_3 v^{(1)} + \dots + n(n-1)a_n v^{(n-2)} \\ \Delta^3 U_v &= 3 \cdot 2 \cdot 1 \cdot a_3 + \dots + n(n-1)(n-2)a_n v^{(n-3)} \\ &\dots \dots \dots \\ \Delta^n U_v &= a_n (n!) \end{aligned} \right\} \quad (154)$$

Since these expressions must hold for all values of x , the coefficients can now be found by setting $v = 0$. Therefore

$$a_0 = U_0, a_1 = \Delta U_0, a_2 = \frac{\Delta^2 U_0}{2!}, \dots, a_n = \frac{\Delta^n U_0}{n!}.$$

Substituting the values for the coefficients into Equation (152), Gregory-Newton's interpolation formula is obtained thus

$$U_v = U_0 + v^{(1)} \Delta U_0 + \frac{v^{(2)}}{2!} \Delta^2 U_0 + \dots + \frac{v^{(n)}}{n!} \Delta^n U_0. \quad (155)$$

Now the value of a function of x may be found by replacing the function with this interpolating polynomial, then differentiating

$$\frac{df(\bar{x})}{dx} \cong \frac{dU_{\bar{x}}}{dx} = \frac{dU_v}{dv} \frac{dv}{dx} \bigg|_{x=\bar{x}}. \quad (156)$$

The derivative of Equation (153) gives

$$\frac{dv}{dx} = \frac{1}{h} \quad (157)$$

Substituting Equation (157) into Equation (156) gives the derivative

$$\frac{df(\bar{x})}{dx} \cong \frac{1}{h} \frac{dU_v}{dv} \quad (158)$$

An approximation of the second derivative of $f(\bar{x})$ can be found by again differentiating Equation (156)

$$\frac{d^2 f(\bar{x})}{dx^2} \cong \frac{d^2 U_v}{dv^2} = \frac{d}{dv} \left[\frac{1}{h} \frac{dU_v}{dv} \right] \frac{dv}{dx} \quad (159)$$

Recalling that $\frac{dv}{dx} = \frac{1}{h}$, Equation (159) can be rewritten

$$\frac{d^2 f(\bar{x})}{dx^2} \cong \frac{1}{h^2} \frac{d^2 U_v}{dv^2} \quad (160)$$

Applying the expression relating the derivative of the function to the derivative of the interpolation formula (155) gives the first derivative

$$\frac{du}{dx} \cong \frac{1}{h} \left[\frac{dU_v}{dv} \right]_{v=0} = \frac{1}{h} \left(\Delta U_0 - \frac{1}{2} \Delta^2 U_0 + \frac{1}{3} \Delta^3 U_0 - \frac{1}{4} \Delta^4 U_0 + \dots \right) \quad (161)$$

Clearly, the first approximation is

$$\left(\frac{du}{dx} \right)_{x=x_0} \cong \frac{\Delta U_0}{h} = \frac{1}{h} (U_1 - U_0)$$

or in general

$$\left(\frac{du}{dx} \right)_{x=i} \cong \frac{1}{h} (U_{i+1} - U_i) \quad (162)$$

This approximation is most usefully applied to the boundary condition where the derivative does not exist for the $i-1$ element.

The finite difference approximation for the time derivative is found in a similar manner resulting in

$$\left(\frac{du}{dt} \right)_{t=t_0} \cong \frac{1}{p} (U_{m+1} - U_m) \quad (163)$$

Here p is the difference between two successive values of the argument t .

Another interpolation formula can be developed by averaging the forward and backward interpolation formulas of Gauss. The forward interpolation formula expresses the temperature

$$u = U_0 + v \Delta U_{1/2} + \frac{1}{2!} v^{(2)} \Delta^2 U_0 + \frac{1}{3!} (v+1)^{(3)} \Delta^3 U_{1/2} + \dots + \frac{1}{(2n-1)!} (v+n-1)^{(2n-1)} \Delta^{2n-1} U_{1/2} + \frac{1}{(2n)!} (v+n-1)^{(2n)} \Delta^{2n} U_0 + \dots \quad (164)$$

In the backward interpolation formula

$$u = U_0 + v \Delta U_{-1/2} + \frac{1}{2!} (v+1)^{(2)} \Delta^2 U_0 + \frac{1}{3!} (v+1)^{(3)} \Delta^3 U_{-1/2} + \dots + \frac{1}{(2n)!} (v+n)^{(2n)} \Delta^{2n} U_0 + \frac{1}{(2n+1)!} (v+n)^{(2n+1)} \Delta^{2n+1} U_{-1/2} + \dots \quad (165)$$

Averaging Equations (164) and (165) gives

$$u = U_0 + v \frac{\Delta U_{-1} + \Delta U_0}{2} + \frac{1}{2} v^2 \Delta^2 U_{-1} + \frac{1}{6} v (v^2 - 1) \frac{\Delta^3 U_{-2} + \Delta^3 U_{-1}}{2} + \dots \quad (166)$$

which is the central difference formula of Stirling.

If the relationship (158) is applied to Equation (166), an approximation of the first derivative may be determined

$$\left(\frac{du}{dx} \right)_{x=x_0} \approx \frac{1}{h} \left[\frac{dU_v}{dv} \right]_{v=0} = \frac{1}{h} \left(\frac{\Delta U_{-1} + \Delta U_0}{2} - \frac{1}{6} \frac{\Delta^3 U_{-2} + \Delta^3 U_{-1}}{2} + \dots \right) \quad (167)$$

Again, using the first approximation gives

$$\left(\frac{du}{dx} \right)_{x=x_0} \approx \frac{\Delta U_{-1} + \Delta U_0}{2h} = \frac{U_1 - U_{-1}}{2h} \quad (168)$$

Now the derivative squared term is obtained simply by squaring the right-hand member of Equation (168)

$$\left(\frac{du}{dx} \right)_{x=x_0}^2 = \frac{1}{4h^2} (U_1^2 - 2U_1 U_{-1} + U_{-1}^2) \quad (169)$$

From Equation (166), the first approximation of the second derivative is

$$\begin{aligned} \left(\frac{d^2 u}{dx^2} \right)_{x=x_0} &\cong \frac{1}{h^2} \left[\frac{d^2 U_v}{dv^2} \right]_{v=0} = \frac{1}{h^2} (\Delta^2 U_{-1}) \\ &\cong \frac{1}{h^2} \Delta (\Delta U_{-1}) = \frac{1}{h^2} [(U_1 - U_0) - (U_0 - U_{-1})] \\ \left(\frac{d^2 u}{dx^2} \right)_{x=x_0} &\cong \frac{1}{h^2} (U_1 - 2U_0 + U_{-1}) . \end{aligned} \quad (170)$$

This completes the derivation of all finite difference equations that are required to solve the general heat-transfer equation and the boundary conditions in Section 3.

If the heat diffusion Equation (13) could be solved conveniently by a Fourier series expansion, we could show that stability of the solution depends on the ratio

$$M = \frac{k p}{\rho c_p h^2} = \frac{\alpha p}{h^2} \quad (171)$$

Since this would be somewhat lengthy to show, it will just be noted that an examination of this parameter is required prior to attempting a solution of the problem. If, for instance, $M < 1/2$, each term in the solution of the difference equation will decrease exponentially with time. This means that round off error, although carried to subsequent rows, would eventually vanish. On the other hand, if $M > 1/2$, some terms would increase exponentially with time, and round off errors would tend to be amplified as the solution proceeds. Therefore, once an h has been selected, it becomes important to choose a value for p such that $M < 1/2$ to insure stability of the solution.

APPENDIX II

METHOD OF LEAST SQUARES

This section reviews a method of determining the constants that appear in the equation that is selected to represent thermal conductivity and specific heat data. It is the most useful method and the one most frequently applied since it has the advantage of producing a unique set of values for the constants of the polynomials. Moreover, these constants give the most probable equation in the sense that the computed values of $k(u)$ or $c_p(u)$ are the most probable values of the observations, since the residuals are assumed to follow the Gaussian law of error. In other words, the principle of the method of least squares asserts that the most representative curve is that for which the sum of the squares of the residuals is a minimum.

Suppose the given set of observed values (k_i, u_i) , $(i = 1, 2, 3, \dots, n)$, can be represented by the equation

$$k = f(u) \quad (172)$$

containing r undetermined constants, A_1, A_2, \dots, A_r . Then the n^{th} observation equations, for example

$$k_i = f(u_i),$$

are to be solved for the r unknowns. If $r = n$ there are just enough conditions to determine the constants; if $n < r$, there are not enough conditions and the problem is indeterminate; but, in general, $n > r$, and there are more conditions than there are unknowns. In the general case, the values of an m which satisfy any r of these equations will not satisfy the remaining $n - r$ equations, and the problem is to determine the set of values of a_m that will give the most probable values of k . Let

$$v_i = \bar{k}_i - k \quad (173)$$

be the residuals or deviations of the computed values from the observed values, where \bar{k}_i is the value of k obtained by substituting $u = u_i$ in $k = f(u)$. On the basis of the Gaussian law of error, the probability of obtaining the observed values k_i is

$$P = \left(\frac{h}{\sqrt{\pi}} \right)^n e^{-h^2 \sum_{i=1}^n v_i^2} \quad (174)$$

P is a maximum where $\sum_{i=1}^n v_i^2$ is a minimum. Since $s \equiv \sum_{i=1}^n v_i^2$ is a function of the r unknowns, A_1, A_2, \dots, A_r , it follows that the necessary conditions for a minimum are

$$\frac{\partial s}{\partial A_1} = 0, \quad \frac{\partial s}{\partial A_2} = 0, \quad \dots, \quad \frac{\partial s}{\partial A_r} = 0. \quad (175)$$

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Moreover, each v_i is a function of a_m ; therefore,

$$\frac{\partial s}{\partial A_m} \equiv \frac{\partial}{\partial A_m} (v_1^2 + v_2^2 + \dots + v_n^2)$$

or

$$\begin{aligned} \frac{\partial s}{\partial A_1} &= 2v_1 \frac{\partial v_1}{\partial A_m} + 2v_2 \frac{\partial v_2}{\partial A_m} + \dots + 2v_n \frac{\partial v_n}{\partial A_m}, \\ \frac{\partial s}{\partial A_1} &= 2 \sum_{i=1}^n v_i \frac{\partial v_i}{\partial A_m}, \quad (m = 1, 2, \dots, r). \end{aligned} \quad (176)$$

These formulas, Equation (175), are called the normal equation.

If the r functions in Equation (175) are linear in the r unknown A_1, A_2, \dots, A_r , then these equations can be solved immediately. This will certainly be the case if $f(u)$ is a polynomial. Let

$$f(u) = \sum_{j=1}^r A_j u^{j-1} \quad (177)$$

so that

$$v_i = \sum_{j=1}^r A_j u_i^{j-1} - k_i. \quad (178)$$

Then $\frac{\partial v_i}{\partial A_m} = u_i^{m-1}$, and the normal equations assume the form, with the aid of Equation (176),

$$\sum_{i=1}^n \left(\sum_{j=1}^r A_j u_i^{j-1} - k_i \right) u_i^{m-1} = 0, \quad (m = 1, 2, \dots, r). \quad (179)$$

One should note that the equation that is obtained by setting $m = 1$ is

$$\sum_{i=1}^n v_i = 0 \quad (180)$$

Rearranging the terms in Equation (180) and collecting the coefficients of A_j gives

$$\sum_{j=1}^r \left(\sum_{i=1}^n u_i^{j+m-2} \right) A_j = \sum_{i=1}^n u_i^{m-1} k_i \quad (m = 1, 2, \dots, r). \quad (181)$$

Now the r linear equations can be solved for the values of the r unknowns $A_1, A_2, A_3, \dots, A_r$. For example:

U	K
1	1.7
2	1.8
3	2.3
4	3.2

Now express $f(u) = A_1 + A_2 u + A_3 u^2$, then $v_i = A_1 + A_2 u_i + A_3 u_i^2 - k_i$, and

$$\frac{\partial v_i}{\partial A_1} = 1, \quad \frac{\partial v_i}{\partial A_2} = u_i, \quad \frac{\partial v_i}{\partial A_3} = u_i^2$$

The normal equations are

$$\frac{\partial s}{\partial A_m} = 2 \sum_{i=1}^n v_i \frac{\partial v_i}{\partial A_m} = 0,$$

$$\sum_{i=1}^n v_i \frac{\partial v_i}{\partial A_m} = 0,$$

so that

$$\sum_{i=1}^4 v_i \frac{\partial v_i}{\partial A_m} = 0, \quad (m = 1, 2, 3)$$

or in other terms

$$\sum_{i=1}^4 (A_1 + A_2 u_i + A_3 u_i^2 - k_i) \cdot 1 = 0,$$

$$\sum_{i=1}^4 (A_1 + A_2 u_i + A_3 u_i^2 - k_i) u_i = 0,$$

$$\sum_{i=1}^4 (A_1 + A_2 u_i + A_3 u_i^2 - k_i) u_i^2 = 0$$

If the coefficients of A_j are collected and the normal equations put in the form of Equation (182), the following three equations result:

$$4A_1 + \left(\sum_{i=1}^4 u_i \right) A_2 + \left(\sum_{i=1}^4 u_i^2 \right) A_3 = \sum_{i=1}^4 k_i,$$

$$\left(\sum_{i=1}^4 u_i \right) A_1 + \left(\sum_{i=1}^4 u_i^2 \right) A_2 + \left(\sum_{i=1}^4 u_i^3 \right) A_3 = \sum_{i=1}^4 u_i k_i,$$

$$\left(\sum_{i=1}^4 u_i^2 \right) A_1 + \left(\sum_{i=1}^4 u_i^3 \right) A_2 + \left(\sum_{i=1}^4 u_i^4 \right) A_3 = \sum_{i=1}^4 u_i^2 k_i .$$

Now

$$\sum_{i=1}^4 u_i = 1 + 2 + 3 + 4 = 10 ,$$

$$\sum_{i=1}^4 u_i^2 = 1 + 4 + 9 + 16 = 30 , \text{ etc.}$$

so that

$$4 A_1 + 10 A_2 + 30 A_3 = 9 ,$$

$$10 A_1 + 30 A_2 + 100 A_3 = 25 ,$$

$$30 A_1 + 100 A_2 + 354 A_3 = 80.8 .$$

When these equations are solved for the coefficients,

$$A_1 = 2 , \quad A_2 = -0.5 , \quad A_3 = 0.2 .$$

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Aeronautical Systems Division, Dyna-Soar Engineering Office, Booster and Power Division, Wright-Patterson Air Force Base, Ohio.
Rpt Nr ASD-TDR-62-625, PROCEDURES FOR THE DESIGN OF THERMAL PROTECTION SYSTEMS FOR MANEUVERABLE RE-ENTRY VEHICLES, Sep 62, 73p. Incl illus, tables, 17 refs.

Unclassified Report

Atmospheric re-entry of earth-orbital, hypersonic glide vehicles creates thermal problems. The heat affects not only the materials and construction of the airframe but also the crew and various subsystems of the vehicle. Successful solution of these problems depends upon the development of an effective thermal protective concept, which will also give the designer some latitude in his design philosophy. The role of the

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protective system is to significantly attenuate the influx of heat that is aerodynamically generated within the surrounding boundary layer. Attenuation is accomplished by combining external radiation shielding elements with backup insulation materials and an appropriate cooling system.

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